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REVIEW ARTICLES

The Serret–Andoyer Formalism in Rigid-Body Dynamics: II. Geometry, Stabilization, and Control

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Abstract—This paper continues the review of the Serret–Andoyer (SA) canonical formalism in rigid-body dynamics, commenced by [1], and presents some new results. We discuss the applications of the SA formalism to control theory. Considerable attention is devoted to the geometry of the Andoyer variables and to the modeling of control torques. We develop a new approach to Stabilization of rigid-body dynamics, an approach wherein the state-space model is formulated through sets of canonical elements that partially or completely reduce the unperturbed Euler–Poinsot problem. The controllability of the system model is examined using the notion of accessibility, and is shown to be accessible. Based on the accessibility proof, a Hamiltonian controller is derived by using the Hamiltonian as a natural Lyapunov function for the closed-loop dynamics. It is shown that the Hamiltonian controller is both passive and inverse optimal with respect to a meaningful performance-index. Finally, we point out the possibility to apply methods of structure-preserving control using the canonical Andoyer variables, and we illustrate this approach on rigid bodies containing internal rotors.

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1. INTRODUCTION

The canonical form of the forced rigid-body-spin problem is ubiquitous in astronomical applications [8–12]. However, most engineering textbooks dealing with rigid-body dynamics do not discuss it. This is true for both classical [13, 14] and more recent [15] textbooks. The only engineering-oriented publication that directly utilized the Andoyer variables was that of Zanardi et al. [16] who adopted a simplified version of the Andoyer formalism for deriving an attitude controller for a fully actuated axisymmetric rigid body, and that of Giacaglia and Jefferys [17], which developed the equations of motion of a space station. Thus, application of the Andoyer variables to Stabilization and control in rigid-body dynamics so far remains a fallow land.

On the other hand, Stabilization of rigid-body dynamics in the *Eulerian* formulation has been extensively studied in the literature. Several successful approaches to the problem, such as the Energy-Casimir method [18], Riemannian geometric mechanics [19], passivity-based control [20] and optimal control [21] have been developed. The control-theoretic problems arising in this context, such as global Stabilization [22], smooth Stabilization [23], output feedback Stabilization [24], robust Stabilization [25], adaptive control [26] and sliding-mode control [27] have been widely explored as well. These works used several variants of kinematic attitude representations [28], but none has utilized the special features offered by the canonical formulation.

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The purpose of this paper will be twofold: We shall provide a coherent portrayal of the Andoyer variables geometry based on the theory of Hamiltonian systems on Lie groups; and we will fill the existing gap in the control literature by using the SA modelling of rigid-body dynamics in order to derive nonlinear asymptotically Stabilizing controllers for an arbitrarily-shaped rigid body. To this end, we shall first present the transformation from the Eulerian variables to the SA formalism, and shall develop a state-space model in terms of the Andoyer variables. We shall also show, following the work of [29], how the Serret—Andoyer transformation may be derived using symplectic (Marsden—Weinstein) reduction. We shall then study the controllability of the rigid-body dynamics in this formulation by using the accessibility rank condition. Based on the controllability analysis, a Hamiltonian controller will be developed by using the Hamiltonian as a natural Lyapunov function for the closed-loop system. It will be demonstrated that the canonical formalism yields a straightforward characterization of important features such as passivity and inverse optimality. Finally, we will demonstrate how the Energy-Casimir method may be used for Stabilization of a rigid body/rotor system about its intermediate axis.

2. GEOMETRY: GENERALITIES IN HAMILTONIAN SYSTEMS WITH SYMMETRY

We now discuss some of the geometry of the Andoyer variables following the work in [29].

We firstly recall in this section some general notions concerning Hamiltonian systems on Lie groups. We shall employ hereafter standard terminology and methodology of geometric methods in control and dynamics, and we recommend, among others, the texts of Abraham and Marsden [2], Arnold [3], and Marsden and Ratiu [7] for in-depth syntheses of the subject. Nevertheless, we shall highlight here some elements of the theory of Hamiltonian systems on $SO(3)$ that are relevant to the development of our subsequent results.

Consider the Lie group $SO(3)$, i.e., the group of real 3×3 orthogonal matrices with determinant equal to 1. Let $\mathfrak{so}(3)$ denote its Lie algebra, and $\mathfrak{so}(3)^*$ its dual. Recall that elements of $\mathfrak{so}(3)$ are 3×3 skew-symmetric matrices, and the algebra bracket is the usual matrix commutator bracket, $[\mathbf{V}, \mathbf{W}] = \mathbf{VW} - \mathbf{WV}$.

As in [1, Sect. 2], let the *hat map* $\widehat{\ }: \mathbb{R}^3 \to \mathfrak{so}(3)$ denote the usual Lie algebra isomorphism that identifies $(\mathfrak{so}(3), [\,,\,])$ with $(\mathbb{R}^3, \times)$:

$$\mathbf{v} = (v_1, v_2, v_3) \ \mapsto \ \widehat{\mathbf{v}} = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix}. \tag{1}$$

By duality, $\mathfrak{so}(3)^*$ is also identified with $\mathbb{R}^3$.

With the above identification, the classical definitions of body and spatial angular velocities $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$, respectively, have the following geometric meaning: the pair $(\mathbf{R}, \widehat{\boldsymbol{\omega}})$ is the *body representation*, i.e., left-translation to the identity, of tangent vectors in $TSO(3)$. Indeed, $\dot{\mathbf{R}} \in T_{\mathbf{R}}SO(3)$ has the form $\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\omega}}$. Thus, left-translation gives $T_{\mathbf{R}}L_{\mathbf{R}}^{-1} \cdot \mathbf{R}\widehat{\boldsymbol{\omega}} = \mathbf{R}^{-1}\mathbf{R}\widehat{\boldsymbol{\omega}} = \widehat{\boldsymbol{\omega}}$. Likewise, the pair $(\mathbf{R}, \widehat{\boldsymbol{\Omega}})$ is the *spatial representation* of tangent vectors by right-translation to the identity. By duality, the body and spatial angular momentum vectors $\mathbf{g}$ and $\mathbf{G}$ give the body and spatial representations, respectively, of covectors in $T^*SO(3)$. The pairing between tangent vectors and covectors is then given by the usual dot product on $\mathbb{R}^3$: $\langle (\mathbf{R}, \mathbf{g}), (\mathbf{R}, \widehat{\boldsymbol{\omega}}) \rangle = \mathbf{g} \cdot \boldsymbol{\omega}$. Arnold [3] gave a clarification of the various representations for general Lie groups, and showed that they can be applied to fluid mechanics. See also [7] for an exposition.

A Hamiltonian function $\mathcal{H}$ on $T^*SO(3)$ is said to be *left-invariant* if $\mathcal{H} \circ L_{\mathbf{R}}^* = \mathcal{H}$ for all $\mathbf{R} \in SO(3)$, where L^* denotes the cotangent lifted action. In the body representation, left-invariance means that $\mathcal{H}$ depends only on the body angular momentum, i.e., $\mathcal{H} : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R} : (\mathbf{R}, \mathbf{g}) \mapsto \mathcal{H}(\mathbf{g})$. The *fibre derivative* of $\mathcal{H}$ is the map $\mathbb{F}\mathcal{H} : SO(3) \times \mathfrak{so}(3)^* \to SO(3) \times \mathfrak{so}(3) : (\mathbf{R}, \mathbf{g}) \mapsto (\mathbf{R}, \nabla_{\mathbf{g}}\mathcal{H})$. $\mathcal{H}$ is said to be *hyperregular* if $\mathbb{F}\mathcal{H}$ is a diffeomorphism. The following is an important lemma, materials for the proof of which can be found in [2, Sect. 4.4].

Lemma 1. *Let $\mathcal{H} : SO(3) \times \mathfrak{so}(3)^* \to \mathbb{R}$ be left-invariant, then the associated Hamiltonian vector field in body coordinates is*

$$X_{\mathcal{H}}(\mathbf{R}, \mathbf{g}) = \left(\mathbf{R} \cdot \widehat{\nabla_{\mathbf{g}}\mathcal{H}}\,, \;\; \mathbf{g} \times \nabla_{\mathbf{g}}\mathcal{H}\right). \tag{2}$$

Moreover, if $\mathcal{H}$ is hyperregular, the associated left-invariant Lagrangian $\mathcal{L} : SO(3) \times \mathfrak{so}(3) \to \mathbb{R}$ is given in body coordinates by

$$\mathcal{L}(\boldsymbol{\omega}) = \mathbf{g} \cdot \boldsymbol{\omega} - \mathcal{H}(\mathbf{g}), \tag{3}$$

where $\mathbf{g}$ is given in terms of $\boldsymbol{\omega}$ by the Legendre transform $\mathbb{F}\mathcal{L} : SO(3) \times \mathfrak{so}(3) \to SO(3) \times \mathfrak{so}(3)^*$: $(\mathbf{R}, \boldsymbol{\omega}) \mapsto (\mathbf{R}, \mathbf{g}) = (\mathbf{R}, \nabla_{\boldsymbol{\omega}}\mathcal{L})$, with $(\mathbb{F}\mathcal{L})^{-1} = \mathbb{F}\mathcal{H}$.

The second component of $X_{\mathcal{H}}$ is sometimes called the *Euler vector field*, and the dynamical equations

$$\dot{\mathbf{g}} = \mathbf{g} \times \nabla_{\mathbf{g}}\mathcal{H} \tag{4}$$

are called *Euler–Poinsot equations*. In particular, the classical Lagrangian of free rigid body dynamics is given in body coordinates as $\mathcal{L}(\boldsymbol{\omega}) = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbb{I}\boldsymbol{\omega}$. Thus, the body angular momentum is the image by the Legendre transform of the body angular velocity, $\mathbb{F}\mathcal{L} : \boldsymbol{\omega} \mapsto \mathbf{g} = \mathbb{I}\boldsymbol{\omega}$, and the classical spatial angular momentum is its spatial representation.

In general, given a hyperregular Hamiltonian $\mathcal{H}$ (or, equivalently, a hyperregular Lagrangian $\mathcal{L}$), one can define a generalized body angular momentum by the Legendre transform, i.e., $\mathbf{g} = \nabla_{\boldsymbol{\omega}}\mathcal{L}$, where $\boldsymbol{\omega}$ is the classical body angular velocity. $\mathcal{H}$ in our setting will generally be different from the classical Hamiltonian. For example, for the controlled rigid body, $\mathcal{H}$ can be called the *controlled Hamiltonian*, and $\mathcal{L}$ the *controlled Lagrangian*, to be discussed in Section 6.

We conclude this section with the following property related to the 3-1-3 Euler angle representation. This property is crucial for the generalization of the Serret–Andoyer transformation discussed in the next section.

Lemma 2. *Let $\mathcal{H} \in \mathcal{F}(SO(3) \times \mathfrak{so}(3)^*)$ be a left-invariant, hyperregular Hamiltonian, and let $\mathcal{L}$ be the associated Lagrangian. Then, choosing any arbitrary spatial frame and the set (ϕ, θ, ψ) of 3-1-3 Euler angles, the conjugate momenta (Φ, Θ, Ψ) associated with $\mathcal{L}$ are related to the covector body representation, for all $(\mathbf{R}, \mathbf{g}) \in SO(3) \times \mathfrak{so}(3)^*$, by*

$$\Phi = (g_1 \sin\psi + g_2 \cos\psi)\sin\theta + g_3 \cos\theta, \tag{5a}$$

$$\Theta = g_1 \cos\psi - g_2 \sin\psi, \tag{5b}$$

$$\Psi = g_3. \tag{5c}$$

Moreover, in the chosen spatial frame and ignoring the singular points corresponding to $\theta = 0$, the spatial representation, $\mathbf{G} = \mathbf{R}\mathbf{g}$, is then given in terms of these momenta by

$$G_1 = \Theta\cos\phi + \left(\frac{\Psi - \Phi\cos\theta}{\sin\theta}\right)\sin\phi, \tag{6a}$$

$$G_2 = -\Theta\sin\phi - \left(\frac{\Psi - \Phi\cos\theta}{\sin\theta}\right)\cos\phi, \tag{6b}$$

$$G_3 = \Phi. \tag{6c}$$

Proof. By definition, $\Phi = \partial\mathcal{L}/\partial\dot{\phi} = \nabla_{\boldsymbol{\omega}}\mathcal{L} \cdot \mathbf{D}_{\dot{\phi}}\boldsymbol{\omega} = \mathbf{g} \cdot \mathbf{D}_{\dot{\phi}}\boldsymbol{\omega}$. Substituting the expression

$$\boldsymbol{\omega} = \begin{bmatrix} \dot{\theta}\cos\psi + \dot{\phi}\sin\psi\sin\theta \\ -\dot{\theta}\sin\psi + \dot{\phi}\cos\psi\sin\theta \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix} \tag{7}$$

of the body angular velocity into $\Phi = \mathbf{g} \cdot \mathbf{D}_{\dot\phi}\boldsymbol{\omega}$, one easily obtains (5a). Expressions (5b) and (5c) can be similarly obtained. Finally, (6) is obtained by inverting (5) to yield $\mathbf{g}$ in terms of (Φ, Θ, Ψ), and transforming to the spatial frame with $\mathbf{R}$ given by

$$\mathbf{R}(\phi, \theta, \psi) = \begin{bmatrix} \mathrm{c}_\psi \mathrm{c}_\phi - \mathrm{s}_\psi \mathrm{c}_\theta \mathrm{s}_\phi & \mathrm{c}_\psi \mathrm{s}_\phi + \mathrm{s}_\psi \mathrm{c}_\theta \mathrm{c}_\phi & \mathrm{s}_\psi \mathrm{s}_\theta \\ -\mathrm{s}_\psi \mathrm{c}_\phi - \mathrm{c}_\psi \mathrm{c}_\theta \mathrm{s}_\phi & -\mathrm{s}_\psi \mathrm{s}_\phi + \mathrm{c}_\psi \mathrm{c}_\theta \mathrm{c}_\phi & \mathrm{c}_\psi \mathrm{s}_\theta \\ \mathrm{s}_\theta \mathrm{s}_\phi & -\mathrm{s}_\theta \mathrm{c}_\phi & \mathrm{c}_\theta \end{bmatrix}. \tag{8}$$

□

Remark 1. The variables $(\phi, \theta, \psi, \Phi, \Theta, \Psi)$ are a set of local coordinates for $T^*SO(3)$. Hence, Lemma 2 simply relates these coordinates with the vectorial representations, both in the body and in space. Equations (5) and (6) are true for any left-invariant, hyperregular Hamiltonian. In particular, they are true for the free rigid body Hamiltonian.

3. GENERALIZED SERRET–ANDOYER TRANSFORMATION

In this section, we shall reconstruct the Serret–Andoyer transformation by employing the notion of *symplectic (Marsden–Weinstein) reduction*. This notion is essentially based on that of *momentum maps*, which are quantities generated by symmetry (group actions) on a Poisson manifold. By *Noether's theorem*, momentum maps are conserved along the trajectories of a Hamiltonian vector field when the Hamiltonian is itself invariant under the symmetry. The conserved quantity defines a 'slice' of the manifold which, under further assumption of *equivariance*, can be projected onto a smooth manifold, the *reduced phase space*, of lesser dimension equipped with a unique symplectic structure. The trajectories of the original Hamiltonian vector field are thus projected onto those of a reduced Hamiltonian vector field on the reduced phase space. The Serret–Andoyer transformation is none other than the computation in Eulerian coordinates of this process of reduction. In fact, it generalizes to rigid motions with left-invariant, hyperregular Hamiltonians. But, first, we shall introduce materials essential to the discussion.

3.1. Symplectic (Marsden–Weinstein) Reduction of $T^*SO(3)$

Let $\mathcal{G}$ be a Lie group and and let $\mathfrak{g}$ denote its Lie algebra. Moreover, let $\mathcal{P}$ be a *Poisson manifold*, i.e., a manifold with a *Poisson bracket* $\{,\}$ on $\mathcal{F}(\mathcal{P}) = \mathcal{C}^\infty(\mathcal{P})$ such that $(\mathcal{F}(\mathcal{P}), \{,\})$ is a Lie algebra and $\{AB, C\} = \{A, C\}B + A\{B, C\}$ for all A, B and $C \in \mathcal{F}(\mathcal{P})$. Let $\mathcal{G}$ act on $\mathcal{P}$ (on the left) by Poisson maps, $\mathcal{G} \times \mathcal{P} \to \mathcal{P} : (\rho, q) \mapsto L_\rho(q) = \rho \cdot q$, i.e., L_ρ preserves the Poisson bracket for all group element $\rho \in \mathcal{G}$: $\{A, B\} \circ L_\rho = \{A \circ L_\rho, B \circ L_\rho\}$ for all A and $B \in \mathcal{F}(\mathcal{P})$. To this action corresponds an *infinitesimal action* of $\mathfrak{g}$ on $\mathcal{P}$, i.e., the vector field

$$\xi_\mathcal{P}(q) = \left.\frac{d}{dt}\right|_{t=0} \left[e^{t\xi} \cdot q\right], \tag{9}$$

$q \in \mathcal{P}$, $\xi \in \mathfrak{g}$.

Definition 1. *A map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ is called a momentum map if $X_{\langle \mathbf{J}, \xi \rangle} = \xi_\mathcal{P}$ for all $\xi \in \mathfrak{g}$. Moreover, $\mathbf{J}$ is said to be Ad*-equivariant if $\mathbf{J} \circ L_\rho = \mathrm{Ad}^*_\rho \circ \mathbf{J}$ for all $\rho \in \mathcal{G}$.*

The linear map $\mathrm{Ad}^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is the *co-adjoint action* of $\mathcal{G}$ on $\mathfrak{g}^*$ (see [7] for the definition). For $\mathcal{G} = SO(3)$ one has:

Lemma 3. *The co-adjoint action of $SO(3)$ on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ is the usual coordinate transformation in $\mathbb{R}^3$, $\mathrm{Ad}^*_{\mathbf{R}^{-1}}(\mathbf{g}) = \mathbf{R}\mathbf{g}$ for all $\mathbf{R} \in SO(3)$, $\mathbf{g} \in \mathfrak{so}(3)^*$.*

Theorem 1. *Let $\mathcal{P} = T^*\mathcal{G}$ be equipped with the canonical symplectic form and, thus, with the associated Poisson structure. Then, the left–Action of $\mathcal{G}$ on $T^*\mathcal{G}$ is Poisson. Moreover, the Ad*-equivariant momentum mapping of this action is given in body coordinates by $\mathbf{J}(\rho, \mu) = \mathrm{Ad}^*_{\rho^{-1}}(\mu)$ [2, pp. 317–318].*

Note in particular that for $\mathcal{G} = SO(3)$, Lemma 3 and Theorem 1 imply that the associated momentum map is simply the (generalized) angular momentum represented in the inertia 3-space.

Theorem 2 (Noether's Theorem). *If the action of $\mathcal{G}$ on $\mathcal{P}$ is Poisson and admits a momentum map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$, and if the smooth function $\mathcal{H} : \mathcal{P} \to \mathbb{R}$ is $\mathcal{G}$-invariant, i.e., $\mathcal{H} \circ L_\rho = \mathcal{H}$ for all $\rho \in \mathcal{G}$, then $\mathbf{J}$ is a constant of the motion for $X_\mathcal{H}$, i.e., $\mathbf{J} \circ \phi_t = \mathbf{J}$, where ϕ_t is the flow of $X_\mathcal{H}$.*

The following specializes Theorems 1 and 2 to left-invariant Hamiltonian vector fields on cotangent bundles.

Corollary 1. *Let $\mathcal{H} : T^*\mathcal{G} \to \mathbb{R}$ be left-invariant. Then, the spatial representation of the momentum map $\mathbf{J}$ is invariant along the trajectories of $X_\mathcal{H}$.*

Corollary 1 generalizes the classically known fact that the angular momentum of a free rigid body is conserved in space. Now, let $\mathcal{G} = SO(3)$, let $\mathbf{G} \in \mathfrak{so}(3)^*$ be given, and let $\mathcal{M}_\mathbf{G} = \mathbf{J}^{-1}(\mathbf{G})$ denote the *momentum level set* corresponding to $\mathbf{G}$. In this case, $\mathcal{M}_\mathbf{G}$ is a smooth manifold. Since the momentum map $\mathbf{J}$ is equivariant (Theorem 1), the *stationary subgroup* $\mathcal{G}_\mathbf{G} \subset SO(3)$ given by

$$\mathcal{G}_\mathbf{G} = \{\mathbf{R} \in SO(3) : Ad^*_{\mathbf{R}^{-1}}(\mathbf{G}) = \mathbf{G}\}$$

leaves $\mathcal{M}_\mathbf{G}$ fixed.

Proposition 1. *The stationary subgroup $\mathcal{G}_\mathbf{G}$ for the rigid body problem is the 1-parameter subgroup of rotations in the direction of the spatial angular momentum $\mathbf{G}$.*

Proof. Let $\mathbf{R} \in \mathcal{G}_\mathbf{G}$. Then, $\mathrm{Ad}^*_{\mathbf{R}^{-1}}(\mathbf{G}) = \mathbf{G}$ or, equivalently, $\mathbf{RG} = \mathbf{G}$. Hence, $\mathbf{R}$ is a member of the stationary subgroup if and only if $\mathbf{G}$ is an eigenvector of $\mathbf{R}$ (with an eigenvalue of 1), i.e. $\mathbf{R}$ is a rotation in the direction of $\mathbf{G}$. □

The quotient manifold $\mathcal{F}_\mathbf{G} = \mathcal{M}_\mathbf{G}/\mathcal{G}_\mathbf{G}$ is a symplectic manifold endowed with the unique symplectic form $\omega_\mathbf{G}(\alpha, \beta) = \omega(\alpha', \beta')$, where ω is the canonical symplectic form on $T^*SO(3)$, and the vectors α and β tangent to $\mathcal{F}_\mathbf{G}$ at $[x] \in \mathcal{F}_\mathbf{G}$ are obtained by projection of some α' and β' tangent to $\mathcal{M}_\mathbf{G}$ at x [3]. $\mathcal{F}_\mathbf{G}$ is known as the *reduced phase space*. Given a left-invariant Hamiltonian on $T^*SO(3)$, define the *reduced Hamiltonian* $\mathfrak{h}_\mathbf{G} : \mathcal{F}_\mathbf{G} \to \mathbb{R}$ by $\mathcal{H}|_{\mathcal{M}_\mathbf{G}} = \mathfrak{h}_\mathbf{G} \circ \pi_\mathbf{G}$, where $\pi_\mathbf{G}$ is the projection $\pi_\mathbf{G} : \mathcal{M}_\mathbf{G} \to \mathcal{F}_\mathbf{G}$. Then, the trajectories of $X_\mathcal{H}$ project to those of $X_{\mathfrak{h}_\mathbf{G}}$. One therefore obtains, as an image by reduction of the original Hamiltonian system, another Hamiltonian system on the reduced phase space with the above-mentioned symplectic structure.

By Lemma 2, given $\mathcal{H}$ left-invariant and hyperregular, $\mathcal{M}_\mathbf{G}$ can be characterized in Eulerian coordinates by the values of $(\phi, \theta, \psi, \Phi, \Theta, \Psi)$ satisfying (6) for $\mathbf{G}$ fixed. Moreover, to factor out the action of $\mathcal{G}_\mathbf{G}$ on $\mathcal{M}_\mathbf{G}$, one recalls that ϕ is ignorable in $\mathcal{H}$ (see Eq. (23) in [1]) for an arbitrarily chosen spatial frame. We may always choose a spatial frame such that the axis $\mathbf{s}_3$ is parallel to $\mathbf{G}$. In other words, consider $\mathbf{G} \cong (0, 0, G)$ where $G \in \mathbb{R}$ is a nonzero constant. Substituting into (6) and ignoring the singular points corresponding to $\theta = 0$ yields the following result.

Proposition 2. *Let $\mathcal{H} \in \mathcal{F}(SO(3) \times \mathfrak{so}(3)^*)$ be a left-invariant, hyperregular Hamiltonian, and let $\mathbf{G} \in \mathfrak{so}(3)^*$ be fixed. Choose a spatial frame in which $\mathbf{G} \cong (0, 0, G)$, G being a nonzero constant. Relative to this spatial frame, denote the 3-1-3 Euler angles by (ϕ, θ, l) and their conjugate momenta by (Φ, Θ, L). Then the momentum level set $\mathcal{M}_\mathbf{G}$ is locally given by*

$$\Theta = 0, \quad \cos\theta = L/G, \quad \Phi = G, \tag{10}$$

$(\phi, l, L) \in (0, 2\pi) \times (-\pi, \pi) \times (-G, G)$. *Moreover, the map $\pi_\mathbf{G} : \mathcal{M}_\mathbf{G} \to \mathcal{F}_\mathbf{G}$ is the coordinate projection $(\phi, l, L) \mapsto (l, L)$.*

Proof. By the above choice of spatial frame, (6c) yields immediately $\Phi = G$. Renaming (ψ, Ψ) with (l, L), (6a) and (6b) therefore yield $\Theta = 0$ and $L = G\cos\theta$ since $G_1 = G_2 = 0$. This shows that $\mathcal{M}_\mathbf{G}$ is a 3-dimensional manifold with local coordinates (ϕ, l, L). Finally, by Proposition 1, elements of the stationary subgroup $\mathcal{G}_\mathbf{G}$ are rotations of angle ϕ about $\mathbf{s}_3$ leaving the variables (l, L) fixed. This shows that $\pi_\mathbf{G}$ is the coordinate projection along ϕ. □

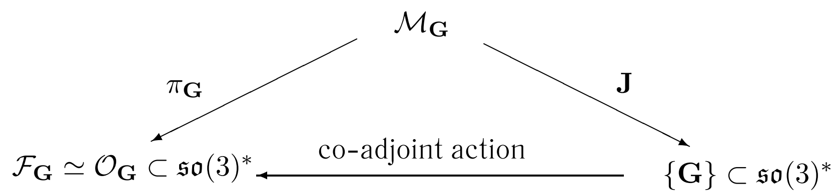
Remark 2. Geometrically, the Andoyer variables (l, L) are the third direction cosine and conjugate momentum in the 3-1-3 Euler angle representation associated with the specially chosen spatial frame, as we mentioned before; in fact, one easily verifies that (l, L) are exactly the same geometric objects as those, by the same names, that result from the classical Serret–Andoyer transformation described in [1, §2]. Moreover, (10) reproduces the classical transformation.

3.2. The Andoyer Variables as Canonical Coordinates of the Co-Adjoint Orbit

The reduced phase space $\mathcal{F}_{\mathbf{G}}$ can be identified with the orbit $\mathcal{O}_{\mathbf{G}}$ of $\mathbf{G} \in \mathfrak{so}(3)^*$ under the co-adjoint action [3]. Indeed, the projection $\pi_{\mathbf{G}}$ has the following expression in body coordinates:

$$\pi_{\mathbf{G}}(\mathbf{R}, \mathbf{g}) = \mathrm{Ad}^*_{\mathbf{R}} \circ \mathbf{J}(\mathbf{R}, \mathbf{g}) = \mathrm{Ad}^*_{\mathbf{R}} \mathbf{G} = \mathbf{g},$$

for all $(\mathbf{R}, \mathbf{g}) \in \mathcal{M}_{\mathbf{G}}$. Hence, $\mathcal{O}_{\mathbf{G}}$ is the body representation of $\mathcal{M}_{\mathbf{G}}$. Schematically, this can be represented by



More precisely, $\mathcal{O}_{\mathbf{G}}$ is given by

$$\mathcal{O}_{\mathbf{G}} = \{\mathbf{g} \in \mathbb{R}^3 : \mathbf{g} = \mathbf{R}^{-1}\mathbf{G}, \;\; \mathbf{R} \in SO(3)\} = \{\mathbf{g} \in \mathbb{R}^3 : \|\mathbf{g}\| = \|\mathbf{G}\|\},$$

i.e., $\mathcal{O}_{\mathbf{G}}$ is the sphere traced by body angular momentum vectors that have magnitude $\|\mathbf{G}\|$, classically known as the *momentum sphere*. $\mathcal{O}_{\mathbf{G}}$ is a symplectic manifold with the unique symplectic forms $\omega^{\pm}$, called *Kostant–Kirillov* symplectic forms, given by

$$\omega^{\pm}_{\mathbf{g}}(\widehat{\mathbf{v}}_{\mathfrak{so}(3)^*}(\mathbf{g}), \widehat{\mathbf{w}}_{\mathfrak{so}(3)^*}(\mathbf{g})) = \pm \mathbf{g} \cdot (\mathbf{v} \times \mathbf{w}), \tag{11}$$

$\mathbf{g} \in \mathcal{O}_{\mathbf{G}}$, $\mathbf{v}$, $\mathbf{w} \in \mathbb{R}^3$, where tangent vectors to $\mathcal{O}_{\mathbf{G}}$ have the form $\widehat{\mathbf{v}}_{\mathfrak{so}(3)^*}(\mathbf{g}) = \mathbf{v} \times \mathbf{g}$. See [7] for an exposition; see also [3].

Proposition 3. *The Andoyer variables* (l, L) *given in Proposition 2 define a local chart* $\mathcal{U}_{\mathbf{G}} : (-\pi, \pi) \times (-G, G) \to \mathcal{O}_{\mathbf{G}}$ *given by*

$$\mathbf{g} = \mathcal{U}_{\mathbf{G}}(l, L) = \left(\sqrt{G^2 - L^2}\sin l, \; \sqrt{G^2 - L^2}\cos l, \; L\right). \tag{12}$$

Moreover, (l, L) *are canonical coordinates with respect to the left* $(-)$ *Kostant–Kirillov symplectic form.*

Proof. As remarked above, $\mathcal{O}_{\mathbf{G}}$ is the body representation of $\mathcal{M}_{\mathbf{G}}$. Substituting (10), which defines $\mathcal{M}_{\mathbf{G}}$, into the expression for body angular momentum, yields (12) after eliminating θ and renaming (ψ, Ψ) with (l, L). Differentiating (12) yields

$$\dot{g}_1 = -\frac{\sin l}{\sqrt{G^2 - L^2}} L\dot{L} + \sqrt{G^2 - L^2}\cos l\,\dot{l} = -\frac{\sin l}{\sqrt{G^2 - L^2}}\dot{L}g_3 + \dot{l}g_2, \tag{13}$$

$$\dot{g}_2 = -\frac{\cos l}{\sqrt{G^2 - L^2}} L\dot{L} - \sqrt{G^2 - L^2}\sin l\,\dot{l} = -\frac{\cos l}{\sqrt{G^2 - L^2}}\dot{L}g_3 - \dot{l}g_1, \tag{14}$$

$$\dot{g}_3 = \dot{L}, \tag{15}$$

i.e., tangent vectors to $\mathcal{O}_{\mathbf{G}}$ have the form $\dot{\mathbf{g}} = \mathbf{v} \times \mathbf{g}$, where

$$\mathbf{v} = \left(\frac{\cos l}{\sqrt{G^2 - L^2}}\dot{L}, \; -\frac{\sin l}{\sqrt{G^2 - L^2}}\dot{L}, \; -\dot{l}\right).$$

Substituting into (11) gives

$$\begin{aligned}\omega^-_{\mathbf{g}}(\mathbf{v}_1\times\mathbf{g},\mathbf{v}_2\times\mathbf{g}) &= -\mathbf{g}\cdot(\mathbf{v}_1\times\mathbf{v}_2)\\ &= -\begin{bmatrix}\sqrt{G^2-L^2}\sin l\\ \sqrt{G^2-L^2}\cos l\\ L\end{bmatrix}\cdot\left(\begin{bmatrix}\frac{\cos l}{\sqrt{G^2-L^2}}\dot L_1\\ -\frac{\sin l}{\sqrt{G^2-L^2}}\dot L_1\\ -\dot l_1\end{bmatrix}\times\begin{bmatrix}\frac{\cos l}{\sqrt{G^2-L^2}}\dot L_2\\ -\frac{\sin l}{\sqrt{G^2-L^2}}\dot L_2\\ -\dot l_2\end{bmatrix}\right)\\ &= \dot l_1\dot L_2-\dot l_2\dot L_1\\ &= dl\wedge dL\left((\dot l_1,\dot L_1),(\dot l_2,\dot L_2)\right),\end{aligned}$$

which proves that (l, L) are canonical local coordinates. □

Remark 3. The Andoyer variables as geometric objects thus have two meanings. On the one hand, they are Eulerian coordinates as mentioned in Remark 2. On the other hand, by Proposition 3, they locally canonically coordinatise the 2-dimensional symplectic manifold $\mathcal{O}_{\mathbf{G}}$. In particular, (l, L) can be viewed as the 'longitude' and 'latitude' on the momentum sphere. In addition, the canonical symplectic form can be viewed as the area element $dl\wedge dL$ that is oriented inward, which corresponds to the $(-)$ sign of the left Kostant–Kirillov symplectic form.

3.3. Main Result

As an immediate result of Propositions 2 and 3, the Serret–Andoyer transformation can be generalized for any Hamiltonian system on $T^*SO(3)$ with a left-invariant, hyperregular Hamiltonian. Indeed, identifying the reduced phase space with the co-adjoint orbit, the reduced Hamiltonian system lives on the latter. Since the Andoyer variables are canonical coordinates of the co-adjoint orbit, dynamics of the reduced system are thus given in canonical symplectic form in these variables.

Theorem 3 (The generalized Serret–Andoyer transformation). *Let $\mathcal{H}\in\mathcal{F}(SO(3)\times\mathfrak{so}(3)^*)$ be a left-invariant, hyperregular Hamiltonian, and let $\mathbf{G}\in\mathfrak{so}(3)^*$ be the conserved spatial momentum. Under the conditions of Proposition 2, the reduced Hamiltonian $\mathfrak{h}_{\mathbf{G}}$ is locally given in the Andoyer variables by*

$$\mathfrak{h}_{\mathbf{G}}(l,L)=\mathcal{H}\circ\mathcal{U}_{\mathbf{G}}(l,L), \tag{16}$$

$(l,L)\in(-\pi,\pi)\times(-L,L)$, *where $\mathcal{U}_{\mathbf{G}}$ is defined by* (12). *The reduced dynamics are then given in canonical form, that is, $\dot l=\partial\mathfrak{h}_{\mathbf{G}}/\partial L$, $\dot L=-\partial\mathfrak{h}_{\mathbf{G}}/\partial l$. Moreover, relative to the spatial frame defined in Proposition 2, the integral solution in $SO(3)$ is characterized by the 3-1-3 Euler angles (ϕ,θ,l), with $cos\theta=L/G$ and $\phi=\int\partial\mathcal{H}/\partial\Phi|_{\mathcal{M}_{\mathbf{G}}}dt$.*

Remark 4.

1. The last equality, $\phi=\int\partial\mathcal{H}/\partial\Phi|_{\mathcal{M}_{\mathbf{G}}}dt$, results from the fact that ϕ is ignorable for $\mathcal{H}$ left-invariant. Note that we are taking the restriction of $\partial\mathcal{H}/\partial\Phi$ on the momentum level set, $\mathcal{M}_{\mathbf{G}}$, according to (10). Since $\partial\mathcal{H}/\partial\Phi|_{\mathcal{M}_{\mathbf{G}}}$ is a function solely of (l,L), $\int\partial\mathcal{H}/\partial\Phi|_{\mathcal{M}_{\mathbf{G}}}dt$ is a line integral when $(l(t),L(t))$ are available.

2. The construction leading to Theorem 3 shows that the Serret–Andoyer transformation is the computation in Eulerian coordinates of the symplectic reduction associated with the lifted left–Action of $SO(3)$ on $T^*SO(3)$. In particular, the choice by Serret for the axis $\mathbf{k}$ to be in the direction of the spatial angular momentum yields precisely the conditions of Proposition 2.

3. The result of Theorem 3 provides a reduced representation of the class of systems in question. This representation is given in a two-dimensional phase space which one can think of as the unit circle S^1. This simplifies the numerical integration of the equations of motion to that of S^1 dynamics. One then reconstructs the full dynamics on $T^*SO(3)$ by solving, in closed form, $\theta=\arccos(L/G)$ on the one hand, and taking the line integral $\phi=\int\partial\mathcal{H}/\partial\Phi|_{M_{\mathbf{p}}}dt$ on the other hand.

Example 1 (The classical Serret–Andoyer transformation). *One recovers immediately the results for the free rigid body. Indeed, as mentioned in Remark 2,* (l, L) *are the same geometric objects as encountered in the classical transformation which is then reproduced by the characterization according to* (10) *of the reduced phase space. Finally, substituting the free rigid body Hamiltonian,* $\mathcal{H} = (\mathbb{I}^{-1}\mathbf{g}) \cdot \mathbf{g}$, *into* (16) *for* $\mathbf{g}$ *given by* (12) *yields*

$$\mathcal{H} = \mathfrak{h}_{\mathbf{G}}(l, L) = \frac{1}{2}\left(\frac{s_l^2}{I_1} + \frac{c_l^2}{I_2}\right)(G^2 - L^2) + \frac{L^2}{2I_3}, \tag{17}$$

which is the SA Hamiltonian (cf. [1]).

Theorem 3 generalizes the Serret–Andoyer transformation to a large class of rigid motions other than the usual one. In particular, one can consider rigid bodies subject to control by means of internal torques. The presence of control *a priori* breaks the original symmetry of the phase space, which now consists of $T^*SO(3)$ and the shape space. The basic idea is that, if the control is Hamiltonian and preserves the symmetry on $T^*SO(3)$, so that the motion of the main body or base of the controlled system becomes that of a new (controlled) left-invariant Hamiltonian vector field on $T^*SO(3)$, then Noether's theorem still holds. A controlled momentum vector can then be found that is preserved in space. In the case where the Hamiltonian is also hyperregular, the results of Section 3 can then be applied, yielding a set of Andoyer variables for the controlled motion of the main body.

This methodology will be illustrated in Section 6. In the following section, we embark on our quest for Stabilizing rigid body dynamics using the SA setup. We do not restrict ourselves at this moment to deal with structure-preserving control; this will be dwelt upon in subsequent sections.

4. MODELLING CONTROL INPUTS

The Hamiltonian modelling naturally accommodates control torque inputs. Letting $\mathbf{u}_S$ be a control torque vector *written in SA coordinates*, we can determine its effect on the canonical dynamics by utilizing the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = \mathbf{u}_S \tag{18}$$

where

$$\mathcal{L} = \mathbf{p}^T\dot{\mathbf{q}} - \mathcal{H} = Gg + Hh + Ll - \mathcal{H}. \tag{19}$$

The resulting Hamilton equations, which include the effect of an external torque, are straightforwardly obtained by substitution of (19) into (18),

$$\dot{\mathbf{q}} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \tag{20}$$

$$\dot{\mathbf{p}} = -\frac{\partial \mathcal{H}}{\partial \mathbf{q}} + \mathbf{u}_S. \tag{21}$$

However, Eqs. (20)–(21) are not useful for practical applications, due to the fact that the control torques must be applied in *body* axes. We therefore need to model the effect of the control torque on the body angular rates, and then transform into Andoyer variables. To this end, we shall first note that the *controlled* Euler–Poinsot equations under the effects of a control torque in body axes, $\mathbf{u}$, are given by

$$\dot{\mathbf{g}} = \mathbf{g} \times \nabla_{\mathbf{g}}\mathcal{H} + \mathbf{u}. \tag{22}$$

Substituting the expressions (12) of $\mathbf{g}$ given in Andoyer variables, as well as Eqs. (13)–(15) into (22), and solving for $\dot{l}, \dot{G}, \dot{L}$, yields modified expressions for these derivatives reflecting the effects of $\mathbf{u}$:

$$\dot{l} = L\left(\frac{1}{I_3} - \frac{s_l^2}{I_1} - \frac{c_l^2}{I_2}\right) + \mathbf{w}_1^T\mathbf{u}, \tag{23}$$

$$\dot{G} = \mathbf{w}_2^T\mathbf{u}, \tag{24}$$

$$\dot{L} = (L^2 - G^2)s_l c_l\left(\frac{1}{I_1} - \frac{1}{I_2}\right) + \mathbf{w}_3^T\mathbf{u}, \tag{25}$$

where the vector fields $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are given by

$$\mathbf{w}_1 = \frac{1}{\sqrt{G^2 - L^2}} \begin{bmatrix} \mathrm{c}_l \\ -\mathrm{s}_l \\ 0 \end{bmatrix}, \quad \mathbf{w}_2 = \frac{1}{G} \begin{bmatrix} \mathrm{s}_l\sqrt{G^2 - L^2} \\ \mathrm{c}_l\sqrt{G^2 - L^2} \\ L \end{bmatrix}, \quad \mathbf{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{26}$$

The mapping of control inputs onto the remaining variables, g, h, H, can be found by utilizing the kinematic relations given by [1]. After some algebra, we arrive at the following expressions:

$$\dot{g} = G\left(\frac{s_l^2}{I_1} + \frac{\mathrm{c}_l^2}{I_2}\right) + c_1^T\mathbf{u}, \tag{27}$$

$$\dot{h} = c_2^T\mathbf{u}, \tag{28}$$

$$\dot{H} = c_3^T\mathbf{u}, \tag{29}$$

where

$$c_1 = \frac{1}{G^2\sqrt{G^2 - H^2}} \begin{bmatrix} -\dfrac{LG\mathrm{c}_l\sqrt{G^2 - H^2}}{\sqrt{G^2 - L^2}} - GH\mathrm{c}_l\mathrm{c}_g + HL\mathrm{s}_l\mathrm{s}_g \\ \dfrac{L\mathrm{s}_l\sqrt{G^2 - H^2}}{G\sqrt{G^2 - L^2}} + \dfrac{H\mathrm{s}_l\mathrm{c}_g}{G} + \dfrac{HL\mathrm{c}_l\mathrm{s}_g}{G^2} \\ -\dfrac{H\mathrm{s}_g\sqrt{G^2 - L^2}}{G^2} \end{bmatrix}, \tag{30}$$

$$c_2 = \frac{1}{G\sqrt{G^2 - H^2}} \begin{bmatrix} G\mathrm{c}_l\mathrm{c}_g - L\mathrm{s}_l\mathrm{s}_g \\ -G\mathrm{s}_l\mathrm{c}_g - L\mathrm{c}_l\mathrm{s}_g \\ \sqrt{G^2 - L^2}\mathrm{s}_g \end{bmatrix}, \tag{31}$$

$$c_3 = \frac{\sqrt{G^2 - H^2}}{G^2} \begin{bmatrix} G\mathrm{c}_l\mathrm{s}_g + H\mathrm{s}_l\sqrt{G^2 - L^2}\sqrt{G^2 - H^2} + L\mathrm{s}_l\mathrm{c}_g \\ -G\mathrm{s}_l\mathrm{s}_g + H\mathrm{c}_l\sqrt{G^2 - L^2}\sqrt{G^2 - H^2} + L\mathrm{c}_l\mathrm{c}_g \\ \sqrt{G^2 - L^2}\mathrm{c}_g - LH \end{bmatrix}. \tag{32}$$

In this paper, we shall be solely interested in controlling the Andoyer variables l, G, L. A one-way coupling exists between the kinematics and the dynamics under the SA setup, meaning that the variables l, G, L affect g, h, H but the converse does not hold. The effect of control inputs on the full 6-dimensional state-space will be dwelt upon in a future work.

Consequently, the state vector in our problem is given by $\mathbf{x} = [l, G, L]^T$, and the state space is $\mathcal{X} = \mathbb{S}^1 \times S$, where $S = \mathbb{R}_{\geqslant 0} \times \mathbb{R}$ (in the uncontrolled case, S is the foliation $\{(g_1, g_2, g_3) | g_1^2 + g_2^2 + g_3^2 = G^2\}$). The state-space dynamics can be now written as

$$I(\mathbf{x}, \mathbf{u}) : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + W(\mathbf{x})\mathbf{u} \tag{33}$$

where $\mathbf{f} : \mathcal{X} \to \mathbb{R}^3$ is the vector field

$$\mathbf{f}(\mathbf{x}) = \begin{bmatrix} L\left(\dfrac{1}{I_3} - \dfrac{\mathrm{s}_l^2}{I_1} - \dfrac{\mathrm{c}_l^2}{I_2}\right) \\ 0 \\ (L^2 - G^2)\mathrm{s}_l\mathrm{c}_l\left(\dfrac{1}{I_1} - \dfrac{1}{I_2}\right) \end{bmatrix} \tag{34}$$

and

$$W(\mathbf{x}) = \begin{bmatrix} \dfrac{c_l}{\sqrt{G^2 - L^2}} & -\dfrac{s_l}{\sqrt{G^2 - L^2}} & 0 \\ \dfrac{s_l\sqrt{G^2 - L^2}}{G} & \dfrac{c_l\sqrt{G^2 - L^2}}{G} & \dfrac{L}{G} \\ 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

It is assumed that the control input $\mathbf{u} = [u_1, u_2, u_3]^T$ is *admissible*, viz. a measurable, bounded function $u : [0, \infty) \to \mathbb{R}^3$. As usual, we say that an admissible input u is a *feedback control law* when $\exists K : \mathcal{X} \to \mathbb{R}^3$ such that $u = K(x)$.

The equilibria of I (defined by $\dot{\mathbf{x}} = \mathbf{u} = 0$) are

$$E_1 = \{L_{eq} = 0, l_{eq} = 0, \frac{\pi}{2}, \pi, \ G_{eq} \neq 0\}, \quad E_2 = \{G_{eq} = L_{eq} = 0, l_{eq} \neq 0\}. \tag{36}$$

We observe that for an arbitrary $\mathbf{u}$, the state space representation $I(\mathbf{x}, \mathbf{u})$ is regular for $G \neq L$. Although this singularity seemingly excludes the equilibrium E_1 from the reachable set, it may be avoided by requiring that $\mathbf{u}$ be a smooth feedback controller. We shall expand upon this issue in the next section.

There are a number of insights gained by the SA state-space model presented above. Most importantly, the magnitude of the angular momentum, G, is a controlled state variable. Therefore, $\mathbf{x} \in E_2$ constitutes *detumbling* (three-axis Stabilization). This representation of detumbling via a single state variable constitutes a *reduction* of the Eulerian model, requiring regulation of all three components of the angular velocity.

5. STABILIZING CONTROLLERS

The SA formalism constitutes a convenient framework for developing rigid-body attitude controllers. We shall show that many fundamental results of the rigid body attitude control theory stem naturally from the SA formalism. To that, we shall adopt a somewhat more formal style.

5.1. Accessibility

The first step in synthesizing a controller for any dynamical system is to determine whether the system is controllable. In order to examine controllability properties of the nonlinear system discussed herein, we shall adopt the notion of *accessibility* [32], a weaker form of controllability, defined as follows.

Definition 2. *Let $\mathcal{R}_T(\boldsymbol{x}_0)$ denote the set of states reachable from the initial state $\boldsymbol{x}_0$ in a finite time t_f using some admissible control $\boldsymbol{u} \in \mathcal{U}$. The system I is said to be accessible from $\boldsymbol{x}_0$, if $\mathcal{R}_T(\boldsymbol{x}_0)$ has a nonempty interior in $\mathcal{X}$.*

To use sufficiency conditions for accessibility, we shall also recall the following definitions:

Definition 3. *Δ is a weak accessibility distribution if it is spanned by the Lie algebra $\mathfrak{g}$ generated by f, w_1, w_2, w_3.*

Definition 4. *Δ_0 is a strong accessibility distribution if it is spanned by the Lie ideal $\mathfrak{g}_0 \in \mathfrak{g}$ generated by w_1, w_2, w_3.*

The following well-known theorem provides a sufficient condition for accessibility [32]:

Theorem 4 (accessibility rank condition). *System I is weakly (strongly) accessible from $\boldsymbol{x}_0$ if* $\operatorname{span}\Delta(\boldsymbol{x}_0) = \mathcal{X}$ $(\operatorname{span}\Delta_0(\boldsymbol{x}_0) = \mathcal{X})$.

We can now state the main result regarding accessibility of the rigid-body dynamics modeled by Andoyer variables in the fully-actuated case.

Lemma 4. *System I is accessible $\forall \boldsymbol{x}_0 \in \mathcal{X}$.*

Proof. Interestingly, the SA model yields an *Abelian* Lie algebra, because the Lie brackets vanish, i.e. $[w_1, w_2] = [w_1, w_3] = [w_2, w_3] = 0$. We shall therefore determine weak accessibility of I by calculating the distribution

$$\Delta = \{\mathbf{w}_0,\, \mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3,\, [\mathbf{w}_0, \mathbf{w}_1], [\mathbf{w}_0, \mathbf{w}_2], [\mathbf{w}_0, \mathbf{w}_3]\} \tag{37}$$

where

$$[\mathbf{w}_0, \mathbf{w}_1] = \begin{bmatrix} \dfrac{(a_1 - a_3)L\mathrm{s}_l}{\sqrt{G^2 - L^2}} \\ \dfrac{(a_3 - a_2)L\mathrm{c}_l\sqrt{G^2 - L^2}}{G} \\ (a_1 - a_2)\mathrm{c}_l\sqrt{G^2 - L^2} \end{bmatrix}, \quad [\mathbf{w}_0, \mathbf{w}_2] = \begin{bmatrix} \dfrac{(a_2 - a_3)L\mathrm{c}_l}{\sqrt{G^2 - L^2}} \\ \dfrac{(a_1 - a_3)L\mathrm{s}_l\sqrt{G^2 - L^2}}{G} \\ (a_1 - a_2)\mathrm{s}_l\sqrt{G^2 - L^2} \end{bmatrix}$$

$$[\mathbf{w}_0, \mathbf{w}_3] = \begin{bmatrix} (a_1 - a_3) + (a_2 - a_1)\mathrm{c}_l^2 \\ \dfrac{(a_2 - a_1)(G^2 - L^2)\mathrm{s}_{2l}}{2G} \\ 0 \end{bmatrix}, \quad a_i = 1/I_i,\ i = 1, 2, 3, \tag{38}$$

and showing that[1)]

$$\mathrm{span}\Delta = \mathcal{X}. \tag{39}$$

A calculation of $\mathrm{rank}(\Delta)$ will determine whether (39) is satisfied. To that end, we shall use the identity $\mathrm{rank}\Delta = \mathrm{rank}\Delta\Delta^T$ and examine $\det(\Delta\Delta^T)$. Performing the symbolic calculation entails

$$\begin{aligned} \det(\Delta\Delta^T) = & \mathrm{c}_l^6 \frac{(a_1 - a_2)^3(G^2 - L^2)^3(a_1 + a2 - 2a_3)}{G^2} \\ & - \mathrm{c}_l^4 \frac{(a_1 - a_2)^2(G^2 - L^2)^2 f_1(L, G, a_i)}{G^2} \\ & + \mathrm{c}_l^2 \frac{(a2 - a1)(G^2 - L^2) f_2(L, G, a_i)}{G^2} \\ & + \left[(a_3 - a_1)^2 \left(\frac{L^4}{G^2} - L^2 - 1\right) - \frac{1}{G^2}\right] \\ & \times \left[(a_3^2 + 2a_1a_2 - a_1^2 - 2a_3a_2)\frac{L^2}{G^2} + \frac{1}{G^2} + (a_1 - a_2)^2\right] \end{aligned} \tag{40}$$

where f_1 and f_2 are functions of the state variables satisfying $f_1, f_2 \neq 0\ \forall \mathbf{x} \in \mathcal{X}$, and hence $\det(\Delta\Delta^T) \neq 0$ $\forall \mathbf{x} \in \mathcal{X}$, rendering the system globally accessible. □

A concept related (but, in the nonlinear case, not necessarily identical) to accessibility is that of feedback Stabilizability [33]. The *feedback Stabilization problem* is usually stated as follows: Given some set-point $\mathbf{x}_d \in \mathcal{X}$, find a feedback control law $u \in \mathcal{U}$ that renders $\mathbf{x}_d$ an asymptotically stable equilibrium (i.e. Lyapunov stable and attractive) of I. If such a feedback exits, it is called an *internal asymptotic feedback Stabilizer* of I. If $K : \mathcal{X} \to \mathcal{U} \in \mathcal{C}^\infty$, it is called a *smooth internal asymptotic feedback Stabilizer*. We also distinguish between *local* and *global* feedback Stabilizers.

To design feedback control laws for the attitude dynamics under the SA setup, we shall utilize a Lyapunov-based approach. We will use the fact that the open-loop system is Hamiltonian to find a natural Lyapunov function for the *closed-loop* system, rendering it globally asymptotically stable.

[1)] Higher order distributions yield equivalent results.

5.2. Hamiltonian Lyapunov Controller

In order to derive a *global* asymptotic feedback Stabilizer, i.e. $\forall \mathbf{x} \in \mathcal{X}$, we shall use the fact that the Hamiltonian is a natural Lyapunov function for the closed-loop system, $I(\mathbf{x}, \mathbf{u})$, as it comprises the rotational kinetic energy *only*, which is always non-negative (cf. Eq. (17)). This remarkable feature of the rigid-body dynamics permits a derivation of a simple smooth feedback controller as stated by the following Lemma.

Lemma 5. *Let $K = diag(k_1, k_2, k_3), k_i > 0 \forall i$. If the Lyapunov function $V(\boldsymbol{x}) = \mathcal{H}$, then the smooth control law*

$$\boldsymbol{u} = -KW^T(x)\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} = \begin{bmatrix} -\dfrac{k_1}{I_1}\sqrt{G^2 - L^2}s_l \\ -\dfrac{k_2}{I_2}\sqrt{G^2 - L^2}c_l \\ -\dfrac{k_3}{I_3}Ł \end{bmatrix} \tag{41}$$

is a global smooth asymptotic feedback Stabilizer for $I(\boldsymbol{x}, \boldsymbol{u})$.

Proof. We note that $V = \mathcal{H} > 0 \forall \mathbf{x} \in \mathcal{X} \backslash \{E_1 \cup E_2\}$ and that $\mathcal{H}$ is radially unbounded. Re-writing the equations of motion in Hamiltonian form gives[2)]

$$\dot{\mathbf{x}} = J\frac{\partial H}{\partial \mathbf{x}} + W\mathbf{u}, \tag{42}$$

where J is given by[3)]

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}. \tag{43}$$

Evaluating $\dot{V}(\mathbf{x})$ along the trajectories of I and taking advantage of the fact that the Hamiltonian is a constant of the motion yields

$$\dot{V}(\mathbf{x}) = \left[\frac{\partial H}{\partial \mathbf{x}}\right]^T \dot{\mathbf{x}} = \left[\frac{\partial H}{\partial \mathbf{x}}\right]^T J\frac{\partial H}{\partial \mathbf{x}} + \left[\frac{\partial H}{\partial \mathbf{x}}\right]^T W\mathbf{u} = \left[\frac{\partial H}{\partial \mathbf{x}}\right]^T W\mathbf{u}. \tag{44}$$

Taking

$$u = -K\left[\left(\frac{\partial H}{\partial \mathbf{x}}\right)^T W\right]^T \tag{45}$$

gives $\dot{V} \leqslant 0$. Asymptotic stability stems from LaSalle's invariance principle ($\dot{V} = 0$ if and only if $\mathbf{x} \in E_1 \cup E_2$), and global asymptotic stability emanates from the radial unboundedness of V. □

Remark 5. We observe that the controller (41) is in fact a damping feedback with the Hamiltonian serving as a control Lyapunov function for the closed loop system. This is a unique feature of the Hamiltonian representation of the attitude dynamics.

Remark 6. Equivalently, we could have derived a non-smooth global asymptotic Stabilizer by taking

$$\mathbf{u} = -K\text{sign}\left[W^T(x)\frac{\partial \mathcal{H}}{\partial \mathbf{x}}\right]. \tag{46}$$

However, we shall not dwell upon non-smooth Stabilization in this work.

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2) We model gradients as column vectors for consistency.

3) Note that we are considering only part of the original SA state vector here. If the full state is considered, J becomes the regular orthoskew symplectic matrix

Remark 7. Stabilization about non-zero set-pint, $\mathbf{x}_d$, can be equivalently derived by defining the shifted state $\bar{x} = x - x_d$.

The smooth feedback Stabilizer presented by Eq. (41) is equivalent to a linear state feedback in Eulerian angular velocities, $\mathbf{u} = -K\boldsymbol{\omega}$, which has been previously derived in a number of works [20, 35]. The following propositions outline a few important features of the control law (41).

Proposition 4. *System I with input u and output $(\partial\mathcal{H}/\partial x)^T W$ is passive.*

Proof. Let $V = \mathcal{H}$ be a candidate storage function. Differentiating V along the system trajectories yields Eq. (44). Integrating from 0 to some t_f yields

$$\int_0^{t_f} \left[\frac{\partial H}{\partial \mathbf{x}}\right]^T W\mathbf{u}dt = V(x(T)) - V(x(0)). \tag{47}$$

Since $\mathcal{H} \geqslant 0 \, \forall \mathbf{x} \in \mathcal{X}$ we have

$$\int_0^{t_f} \left[\frac{\partial H}{\partial \mathbf{x}}\right]^T W\mathbf{u}dt + V(x(0)) \geqslant 0, \tag{48}$$

which completes the proof. This result is equivalent to Ref. [20], which used an Eulerian formalism to show passivity.

An additional important property of the SA-based control system is its inverse optimality with respect to a meaningful performance criterion. This a somewhat general property of control Lyapunov function [36]; however, its manifestation in our context under the Hamiltonian formalism is strikingly simple, showing again the usefulness of formulating the rigid-body dynamics in canonical variables. □

Proposition 5. *The control law (41) is inverse optimal with respect to the performance criterion*

$$\mathcal{J} = \int_0^{\infty} (q(x) + u^T u)dt. \tag{49}$$

Proof. The Hamilton–Jacobi–Bellman (HJB) equation for the functional (49) and an optimal return function V^* reads [30]

$$\left(\frac{\partial V^*}{\partial \mathbf{x}}\right)^T f - \frac{1}{4}\left(\frac{\partial V^*}{\partial \mathbf{x}}\right)^T W W^T \frac{\partial V^*}{\partial \mathbf{x}} + q(x) = 0 \tag{50}$$

with the optimal controller being [30]

$$\mathbf{u} = -\frac{1}{2} W^T \frac{\partial V^*}{\partial x}. \tag{51}$$

Let $V^* = \mathcal{H}$. This nullifies the first term on the left hand side of (50). In additions, assume without loss of generality that $K = -1/2I$. Under these conditions,

$$\dot{V}^* = \dot{\mathcal{H}} = \dot{V} = -\frac{1}{2}\mathcal{H}. \tag{52}$$

Hence, choosing $q(x) = \mathcal{H}/2$ implies inverse optimality. □

5.3. Illustrative Example

An important maneuver often encountered in three-axis Stabilized satellites is detumbling, nullifying the initial angular rates resulting from orbital injection. While detumbling using Eulerian angular velocities ($\omega_1 = \omega_2 = \omega_3 = 0$) as state variables does not directly affect the geometry of the body plane relative to the invariable plane, modeling detumbling via the SA setup ($G = L = 0$) permits to control l, the angle between the body $\hat{\mathbf{b}}_1$-axis and the LON $\hat{\mathbf{j}}$, to some desired, not necessarily zero, set-point ($l = l_d \neq 0$). This is an important feature of the SA setup.

To illustrate the detumbling performance of the damping and Hamiltonian controllers, we shall assume that the satellite follows a circular orbit, so the orbital angular velocity is given by

$$\omega_0 = \sqrt{\frac{\mu}{R_0^3}}, \tag{53}$$

where $\mu = 3.986 \cdot 10^5\,\mathrm{km}^2/\mathrm{s}^3$ is the gravitational constant of the Earth, and R_0 is the orbital radius. Let $R_0 = 7000\,\mathrm{km}$, so that $\omega_0 = 0.001078\,\mathrm{rad/s}$. In addition, assume that $I_3 = 1000\,\mathrm{kgm}^2$, $I_2 = 400\,\mathrm{kgm}^2$, $I_3 = 200\,\mathrm{kgm}^2$. The initial conditions chosen were $L_0 = 0.8\,\mathrm{kgm}^2/\mathrm{s}$, $G_0 = 1\,\mathrm{kgm}^2/\mathrm{s}$, $l_0 = 50$ deg. The closed-loop simulation results, exhibiting the performance of the Hamiltonian controller are depicted by Figs. 1 and 2, showing the time history of the states l, G, L and the control torques, respectively. The detumbling maneuver must be completed within a small time window (about 1 s), thus requiring a considerable torque: about 23 Nm. In practice, the actuators will saturate, extending the system settling time.

5.4. Control on the Orbit

The feedback controller given in Section 5.5.2 effectively controls the Andoyer variables (l, G, L), but does not preserve the foliation since the angular momentum is no longer conserved. Alternatively, one can consider control formulations on the orbit space $\mathcal{O}_{\mathbf{G}}$ itself. Indeed, consider the controlled dynamics with the reduced Hamilitonian of (17):

$$\dot{\mathbf{z}} = J\frac{\partial \mathfrak{h}}{\partial \mathbf{z}} + \boldsymbol{\tau}, \quad \mathbf{z} = (l, L) \in \mathcal{X}_z = S^1 \times [-G, G], \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{54}$$

or

$$\dot{l} \;=\; L\left(\frac{1}{I_3} - \frac{\mathrm{s}_l^2}{I_1} - \frac{\mathrm{c}_l^2}{I_2}\right) + \tau_1, \tag{55a}$$

$$\dot{L} \;=\; (L^2 - G^2)\mathrm{s}_l\mathrm{c}_l\left(\frac{1}{I_1} - \frac{1}{I_2}\right) + \tau_2. \tag{55b}$$

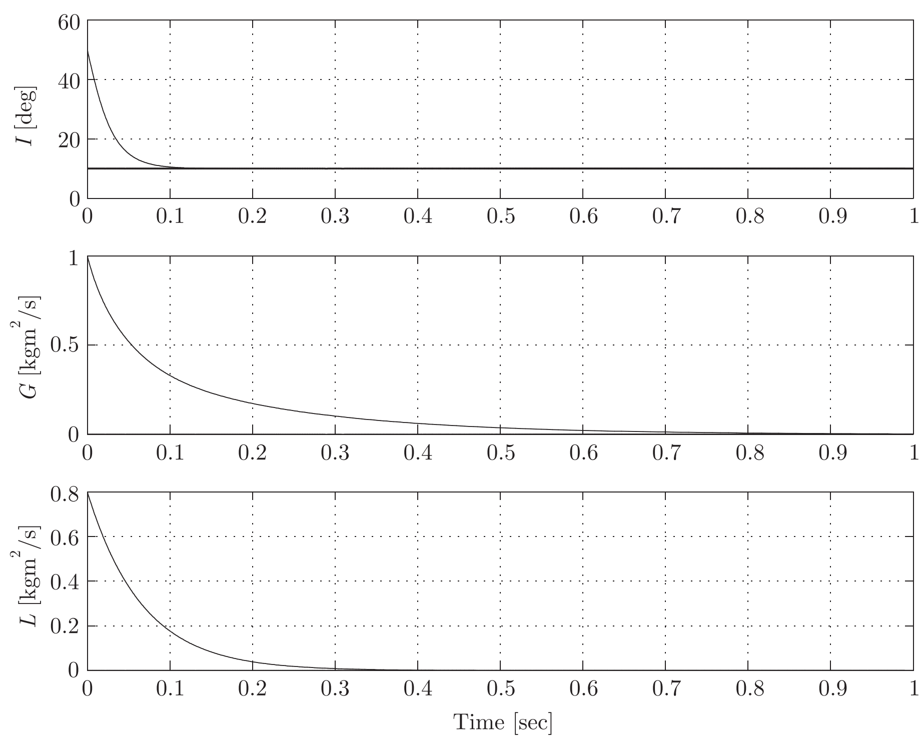

Fig. 1. Detumbling using Hamiltonian control.

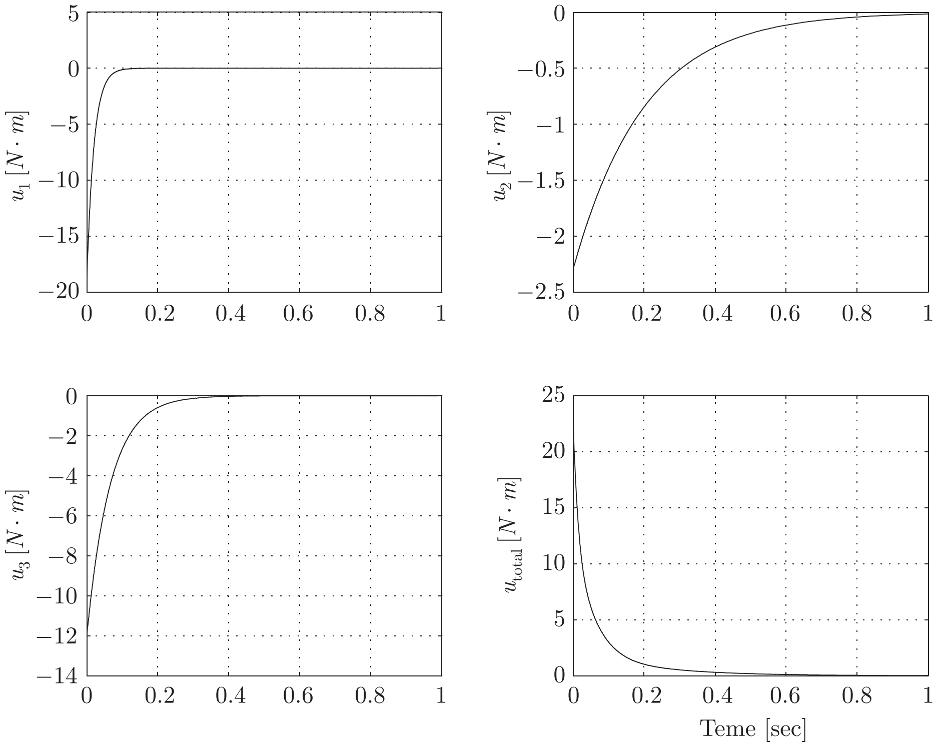

Fig. 2. Detumbling control components and total control torque of the Hamiltonian control.

From the proof of Proposition 3, we can deduce that the control torque $\mathbf{u}$ of the controlled Euler–Poinsot equations (22), which is a vector tangent to $\mathcal{O}_{\mathbf{G}}$, then has the form $\mathbf{u} = \mathbf{v} \times \mathbf{g}$, where

$$\mathbf{v} = \left(\frac{\cos l}{\sqrt{G^2 - L^2}}\tau_2\,,\ -\frac{\sin l}{\sqrt{G^2 - L^2}}\tau_2\,,\ -\tau_1\right). \tag{56}$$

Returning to (54) which is a controlled Hamiltonian system in canonical form, we can similarly adopt the Lyapunov approach of Section 5.5.2. Using $\mathfrak{h}$ as the Lyapunov function, we thus have the following result:

Lemma 6. *Assuming $I_1 > I_2 > I_3$, the smooth control law*

$$\boldsymbol{\tau} = -K\frac{\partial \mathfrak{h}}{\partial \mathbf{z}}, \quad K > 0 \tag{57}$$

globally asymptotically stabilizes spin about the major axis $\mathbf{b}_1$, in the sense that all solutions starting in the compact set $\mathcal{X}_z$ converge to the equilibria

$$E_3 = \{(l, L) : |L| = 0,\ l = \pm\pi/2\}.$$

Proof. With the Lyapunov function $V = \mathfrak{h}$ and the controller (57), $\dot V = (\partial\mathfrak{h}/\partial\mathbf{z})^T K(\partial\mathfrak{h}/\partial\mathbf{z})$. Since $K > 0$, $\dot V = 0$ if and only if $\partial\mathfrak{h}/\partial\mathbf{z} = 0$. It thus follows that the level set $\{\dot V = 0\} = E_3 \cup \{(0,0)\}$. Further analysis of the linearized equations will show that the point $(0,0)$ is an unstable equilibrium, whereas E_3 is a pair of stable equilibria. By LaSalle's invariance principle, all solution starting in $\mathcal{X}_z$ enter E_3. Finally, using (12) one verifies that E_3 corresponds to the body angular momentum $\mathbf{g} = (\pm G, 0, 0)$. This shows that spin about $\mathbf{b}_1$ is globally asymptotically stabilized. □

Remark 8. The controller (57) conserves the magnitude G of the angular momentum by keeping the trajectories on the orbit $\mathcal{O}_{\mathbf{G}}$. However, a quick calculation will show that spatial angular momentum, i.e. the vector $\mathbf{G}$ is not conserved. Hence, strictly speaking it is not a structure-preserving control and, as in the case of the controller (41), the reconstruction of the full dynamics will not be as straight-forward as in the case of the free rigid body.

6. RIGID BODY WITH SINGLE SYMMETRIC ROTOR

We now return to the issue of structure-preserving control, and dwell upon Stabilization by the method of controlled Lagranians — see e.g. [6] and [5].

Consider now a system consisting of a main body (the base) equipped with a single, symmetric rotor aligned with the third principle axis. Let $J_1 = J_2$ and J_3 be the moments of inertia of the symmetric rotor, and denote by γ the angular position of the rotor relative to the body. The Lagrangian of the free system, i.e., in the absence of control, is given by [4]. See also the work [5] and related work on the method of controlled Lagrangians cited in e.g. [6].

$$L_f(\boldsymbol{\omega}, \dot{\gamma}) = \frac{1}{2}\boldsymbol{\omega} \cdot (\mathbb{I} + \mathbb{J})\boldsymbol{\omega} + \frac{1}{2}J_3(\omega_3 + \dot{\gamma})^2, \tag{58}$$

where $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ and $\mathbb{J} = \mathrm{diag}(J_1, J_2, 0)$. The corresponding Legendre's transform is given by

$$\mathbb{F}L_f : \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \dot{\gamma} \end{bmatrix} \longmapsto \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \Gamma \end{bmatrix} = \begin{bmatrix} (J_1 + I_1)\omega_1 \\ (J_2 + I_2)\omega_2 \\ I_3\omega_3 + J_3(\omega_3 + \dot{\gamma}) \\ J_3(\omega_3 + \dot{\gamma}) \end{bmatrix}, \tag{59}$$

which yields the free Hamiltonian

$$\mathcal{H}_f(\mathbf{g}, \Gamma) = \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{(g_3 - \Gamma)^2}{I_3}\right) + \frac{\Gamma^2}{2\,J_3}, \tag{60}$$

where $\lambda_i = I_i + J_i$, $i = 1, 2$.

The configuration space of the present problem is the Lie Group $SO(3) \times S^1$, with Lie algebra $\mathfrak{so}(3)^* \times \mathbb{R}$. The group action in question is $L_{(\mathbf{R},\gamma)}(\mathbf{S}, \varphi) = (\mathbf{RS}, \gamma + \varphi)$. As in the case of the free rigid body (recall Section 2), the rigid body with rotor admits a body representation via the mapping $\bar{\lambda}$: $T^*SO(3) \times T^*S^1 \to (SO(3) \times \mathfrak{so}(3)^*) \times (S^1 \times \mathbb{R}) : (\alpha_{\mathbf{R}}, \gamma, \Gamma) \mapsto (\mathbf{R}, T_e^*L_{\mathbf{R}} \cdot \alpha_{\mathbf{R}}, \gamma, \Gamma)$. As usual, $\mathfrak{so}(3)^*$ is identified with $\mathbb{R}^3$.

Let $\mathcal{H} \in \mathcal{F}(\mathfrak{so}(3)^* \times \mathbb{R}))$, in other words, a left-invariant smooth function on the cotangent bundle. It can be shown, by a result analogous to Lemma 1 for the Lie group $G = SO(3) \times S^1$, that the associated Hamiltonian vector field is given in body coordinates by

$$X_{\mathcal{H}}(\mathbf{R}, \mathbf{g}, \gamma, \Gamma) = \Big(\mathbf{R} \cdot \widehat{\mathbf{D}_1\mathcal{H}}\ ,\ \mathbf{g} \times \mathbf{D}_1\mathcal{H}\ ,\ \partial\mathcal{H}/\partial\Gamma\ ,\ -\partial\mathcal{H}/\partial\gamma\Big), \tag{61}$$

where $\mathbf{D}_1$ denotes the derivative with respect to the first argument. In particular, the equations of motion for the free system are obtained with $\mathcal{H} = \mathcal{H}_f$. In addition, let the system be feedback-controlled by applying a torque $u(\mathbf{R}, \mathbf{g}, \gamma, \Gamma)$ to the rotor, which then yields the following controlled equations of motion.

$$\dot{\mathbf{R}} = \mathbf{R} \cdot \widehat{\mathbf{D}_1\mathcal{H}_f}, \tag{62a}$$

$$\dot{\mathbf{g}} = \mathbf{g} \times \mathbf{D}_1\mathcal{H}_f, \tag{62b}$$

$$\dot{\gamma} = \partial\mathcal{H}_f/\partial\Gamma, \tag{62c}$$

$$\dot{\Gamma} = u(\mathbf{R}, \mathbf{g}, \gamma, \Gamma). \tag{62d}$$

6.1. Structure Preserving Control

Definition 5. *We say that the control $u(\mathbf{R}, \mathbf{g}, \gamma, \Gamma)$ preserves the canonical structure on $T^*SO(3)$ or preserves the rigid body structure if there exists a smooth function $\mathcal{H}_c \in \mathcal{F}(\mathfrak{so}(3)^*)$ such that the closed-loop equations of the base motion have the form*

$$\dot{\mathbf{R}} = \mathbf{R} \cdot \widehat{\nabla_{\mathbf{g}}\mathcal{H}_c}, \tag{63a}$$

$$\dot{\mathbf{g}} = \mathbf{g} \times \nabla_{\mathbf{g}}\mathcal{H}_c. \tag{63b}$$

That is, a control that preserves the rigid body structure yields closed-loop Euler's equation that is Hamiltonian with respect to the usual Lie–Poisson structure on $\mathfrak{so}(3)^*$. Moreover, one can easily verify the following lemma which gives a sufficient condition for such a control.

Lemma 7. *Given the controlled equations of motion* (62)*, a sufficient condition for the control* u *to preserve the rigid body structure is that, along the flow of the closed-loop system,*
i. Γ *is a function of* $\mathbf{g}$*, i.e.,* $\Gamma = \Gamma(\mathbf{g})$*, and*
ii.

$$\nabla_{\mathbf{g}}\mathcal{H}_c(\mathbf{g}) = \mathbf{D}_1\mathcal{H}_f(\mathbf{g}, \Gamma(\mathbf{g})). \tag{64}$$

Bloch et al. [4] gave a Hamiltonian control for which the closed-loop reduced equations are Lie–Poisson on $\mathfrak{so}(3)^*$. In fact, they satisfy (64). We shall prove in the following a slightly more general result.

Proposition 6. *Let* $\varphi : \mathbb{R} \to \mathbb{R}$ *be a* $\mathcal{C}^1$ *function. Then, the feedback control*

$$u(\mathbf{g}) = \varphi'(g_3)\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right) g_1 g_2 \tag{65a}$$

preserves the rigid body structure with the closed-loop Hamiltonian

$$\mathcal{H}_c(\mathbf{g}) = \frac{1}{2}\left(\frac{g_1^2}{\lambda_1} + \frac{g_2^2}{\lambda_2} + \frac{g_3^2}{I_3}\right) - \frac{1}{I_3}\int (\varphi(g_3) + p)\, dg_3, \tag{65b}$$

where p *is a constant.*

Proof. Expanding (62b) and (62c), one gets the following:

$$\dot{m}_1 = \left(\frac{1}{I_3} - \frac{1}{\lambda_2}\right) g_2 g_3 - \frac{1}{I_3}\Gamma g_2, \tag{66a}$$

$$\dot{m}_2 = \left(\frac{1}{\lambda_1} - \frac{1}{I_3}\right) g_1 g_3 + \frac{1}{I_3}\Gamma g_1, \tag{66b}$$

$$\dot{m}_3 = \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) g_1 g_2, \tag{66c}$$

$$\dot{\Gamma} = u. \tag{66d}$$

It can thus be seen that, given (65a), the quantity $p = \Gamma - \varphi(g_3)$ is conserved and, hence, ***i.*** in Lemma 7 is satisfied. Next, by (65b),

$$\nabla_{\mathbf{g}}\mathcal{H}_c(\mathbf{g}) = \left(\frac{g_1}{\lambda_1}, \frac{g_2}{\lambda_2}, \frac{g_3}{I_3} - \frac{1}{I_3}(\varphi(g_3) + p)\right) = \left(\frac{g_1}{\lambda_1}, \frac{g_2}{\lambda_2}, \frac{1}{I_3}(g_3 - \Gamma(g_3))\right) = \mathbf{D}_1\mathcal{H}_f(\mathbf{g}, \Gamma(\mathbf{g})),$$

i.e., (64), which completes the proof.

Remark 9. Setting $\varphi(v) = kv$, where k is a constant, recovers the result in Theorem 5.1 of [4].

6.2. Andoyer Variables for the Control System

By Proposition 6, the base motion of the system subject to the control (65a) is that of a Hamiltonian system on $T^*SO(3)$ with the left-invariant Hamiltonian $\mathcal{H}_c$. The expression of the Hamiltonian (and the associated Lagrangian) depend ultimately on the definition of the function φ. Nevertheless, we are able to proceed implicitly as follows.

Theorem 5. *Suppose that* $\varphi'(v) \neq 1$ *for all* $v \in \mathbb{R}$*. Then, the Hamiltonian* $\mathcal{H}_c$ *given by* (65b) *is hyperregular, and the closed-loop main body motion of the system* (62) *with the control* (65a) *is reduced by the generalized Serret–Andoyer transformation to*

$$\dot{l} = -L\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) + \frac{1}{I_3}\left(L - \varphi(L) - p\right), \tag{67a}$$

$$\dot{L} = (G^2 - L^2)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\sin l \cos l \tag{67b}$$

with the reduced Hamiltonian

$$\mathfrak{h}_{\mathbf{G}}(l,L)=\frac{1}{2}(G^2-L^2)\left(\frac{\sin^2 l}{\lambda_1}+\frac{\cos^2 l}{\lambda_2}\right)+\frac{L^2}{2I_3}-\frac{1}{I_3}\int(\varphi(L)+p)\,dL. \tag{68}$$

Moreover, in the spatial frame defined in Proposition 2, the closed-loop motion of the main body is described by the 3-1-3 Euler angles (ϕ,θ,l), with $cos\theta=L/G$ and $\phi=\int\partial\mathcal{H}_c/\partial\Phi|_{\mathcal{M}_{\mathbf{G}}}dt$.

Proof. We need only to prove that $\mathcal{H}_c$ is hyperregular, providing which the rest of Theorem 5 is a direct application of Theorem 3. By the inverse Legendre transform for the closed loop, i.e., $\boldsymbol{\omega}=\nabla_{\mathbf{g}}H_c$, one gets

$$\omega_1=g_1/\lambda_1, \tag{69a}$$

$$\omega_2=g_2/\lambda_2, \tag{69b}$$

$$\omega_3=(g_3-\varphi(g_3)-p)/I_3. \tag{69c}$$

Since $\varphi'(v)\neq 1$ for all $v\in\mathbb{R}$, the above is invertible with differentiable inverse by the implicit function theorem. Hence, the inverse Legendre transform is a diffeomorphism, i.e., $\mathcal{H}_c$ is hyperregular. □

Example 2. *Let $\varphi(v)=kv$, $k\neq 1$. Then,*

$$\mathfrak{h}_{\mathbf{G}}(l,L)=\frac{1}{2}(G^2-L^2)\left(\frac{\sin^2 l}{\lambda_1}+\frac{\cos^2 l}{\lambda_2}\right)+\frac{((1-k)L-p)^2}{2I_3(1-k)}. \tag{70}$$

In particular, it can be verified that the case where the rotor is locked corresponds to $p=0$ and $k=J_3/(I_3+J_3)$. Substituting the latter expression of k into (70) *recovers the reduced Hamiltonian for the free rigid body with moments of inertia I_i+J_i, $i=1,\ldots,3$.*

6.3. Spin Stabilization about the Intermediate Axis

Suppose in the following that $\lambda_1>\lambda_2>I_3+J_3$, so that the second body axis is the intermediate axis of the locked system. An immediate consequence of Theorem 5 is a simpler stability proof for relative equilibria. First, note that rotation about the intermediate axis, i.e., $\mathbf{g}=(0,G,0)$, corresponds to an equilibrium point at (l,L) in the reduced phase space. In [4], the energy-Casimir method was used to prove stability of the relative equilibria $\mathbf{g}=(0,G,0)$ for the closed-loop Lie–Poisson system. However, for Hamiltonian systems in canonical symplectic form, which is the case of the reduced system (67), the classical Lagrange–dirichlet stability criterion suffices. In effect, the point $(l,L)=(0,0)$ is a stable equilibrium in the sense of Lyapunov if the partial derivatives of $\mathfrak{h}_{\mathbf{G}}$ vanish at $(0,0)$, and if the 2×2 matrix $\delta^2\mathfrak{h}_{\mathbf{G}}$ of second partial derivatives evaluated at $(0,0)$ is either positive- or negative-definite. See [7] for a statement and proof of the Lagrange–dirichlet criterion. The following generalizes Theorem 5.2 in [4].

Theorem 6. *Consider the case $\varphi(0)+p=0$ and $\varphi'(0)>1-I_3/\lambda_2$. Then, the point $(l,L)=(0,0)$ of the reduced system* (67) *is stable in the sense of Lyapunov and, hence, the control* (65a) *Stabilizes rotation about the intermediate axis of the body-rotor system.*

Proof. From (68), $\partial\mathfrak{h}_{\mathbf{G}}/\partial l=0$, and $\partial\mathfrak{h}_{\mathbf{G}}/\partial L=-(\varphi(0)+p)/I_3$ which equals zero if $\varphi(0)+p=0$. The point $(0,0)$ is thus an equilibrium point. Next,

$$\frac{\partial^2\mathfrak{h}_{\mathbf{G}}}{\partial l^2}=(G^2-L^2)\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)(\sin^2 l-\cos^2 l),$$

$$\frac{\partial^2\mathfrak{h}_{\mathbf{G}}}{\partial l\partial L}=2L\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\sin l\cos l,$$

$$\frac{\partial^2\mathfrak{h}_{\mathbf{G}}}{\partial L^2}=\left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\sin^2 l+\left(\frac{1}{I_3}-\frac{1}{\lambda 2}\right)-\frac{1}{I_3}\varphi'(L).$$

Hence,

$$\delta^2 \mathfrak{h}_{\mathbf{G}}(0,0) = \begin{bmatrix} -G^2\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) & 0 \\ 0 & \left(\frac{1}{I_3} - \frac{1}{\lambda 2}\right) - \frac{1}{I_3}\varphi'(0) \end{bmatrix},$$

which is (negative) definite for $\varphi'(0) > 1 - I_3/\lambda_2$. The Lagrange–dirichlet criterion is thus satisfied. □

7. RIGID BODY WITH THREE SYMMETRIC ROTORS

The Serret–Andoyer analysis can also be applied to a system with three rotors. Indeed, consider now the rigid body equipped with three symmetric rotors, each aligned with a principal axis of inertia of the rotor. The Lie group in question is $SO(3) \times S^3$, with $(SO(3) \times \mathfrak{so}(3)^*) \times (S^3 \times \mathbb{R}^3)$ as cotangent bundle in body representation. The Lagrangian of the free (uncontrolled) system is [4]

$$\mathcal{L}_f(\boldsymbol{\omega}, \dot{\gamma}) = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{\Lambda}\boldsymbol{\omega} + \sum_{i=1}^{3} \frac{1}{2} J_i(\omega_i + \dot{\gamma}_i) \tag{71}$$

for $\boldsymbol{\omega} \in \mathfrak{so}(3) \simeq \mathbb{R}^3$, $\dot{\boldsymbol{\gamma}} \in \mathbb{R}^3$, where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ is the locked inertia tensor which is diagonal by the assumption that the rotors are symmetric and aligned with the principal axes. J_i, $i = 1, \ldots, 3$ are the rotors' moments of inertia along their respective axes of rotation. By the Legendre transform, the conjugate momenta are

$$\mathbb{F}\mathcal{L}_f : \begin{bmatrix} \omega_i \\ \dot{\gamma}_i \end{bmatrix} \longmapsto \begin{bmatrix} m_i \\ \Gamma_i \end{bmatrix} = \begin{bmatrix} \lambda_i \omega_i + J_i(\omega_i + \dot{\gamma}_i) \\ J_i(\omega_i + \dot{\gamma}_i) \end{bmatrix}, \quad i \in \{1, \ldots, 3\}, \tag{72}$$

and the free Hamiltonian is left-invariant and is given by

$$\mathcal{H}_f(\mathbf{g}, \boldsymbol{\Gamma}) = \frac{1}{2}\left(\frac{(g_1 - \Gamma_1)^2}{\lambda_1} + \frac{(g_2 - \Gamma_2)^2}{\lambda_2} + \frac{(g_3 - \Gamma_3)^2}{\lambda_3}\right) + \frac{1}{2}\left(\frac{\Gamma_1^2}{J_1} + \frac{\Gamma_2^2}{J_2} + \frac{\Gamma_3^2}{J_3}\right). \tag{73}$$

Introducing control inputs in the form of torques on the rotors, the generic controlled equations are

$$\dot{\mathbf{R}} = \mathbf{R} \cdot \widehat{\mathbf{D}_1 \mathcal{H}_f}, \tag{74a}$$
$$\dot{\mathbf{g}} = \mathbf{g} \times \mathbf{D}_1 \mathcal{H}_f, \tag{74b}$$
$$\dot{\boldsymbol{\gamma}} = \mathbf{D}_2 \mathcal{H}_f, \tag{74c}$$
$$\dot{\boldsymbol{\Gamma}} = \mathbf{u}. \tag{74d}$$

In the following, we consider feedback control of the form $\mathbf{u} : (SO(3) \times \mathfrak{so}(3)^*) \times (S^3 \times \mathbb{R}^3) \to \mathbb{R}^3$.

As in the case of the system with a single rotor, we are interested in controls that preserve the rigid body structure as defined in Definition 5. We recall below a large class of Hamiltonian controls given in [4] that satisfy the conditions of a straightforward extension of Lemma 7 for the present system.

Proposition 7. *Let $\boldsymbol{\varphi} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth map such that the 3×3 matrix $\mathbf{D}\boldsymbol{\varphi}(\mathbf{g})$ is symmetric for all $\mathbf{g} \in \mathbb{R}^3$. Then the feedback controls*

$$\mathbf{u} = \mathbf{D}\boldsymbol{\varphi}(\mathbf{g}) \cdot \dot{\mathbf{g}} \tag{75}$$

for the system (74) *preserve the rigid body structure in the sense of Lemma 7.*

The proof of the above is based on the properties that the given control conserves the quantity $\boldsymbol{\Gamma} - \boldsymbol{\varphi}(\mathbf{g})$, and that the symmetry condition guarantees that there exists an $\mathcal{H}_c(\mathbf{g})$ such that $\nabla_{\mathbf{g}}\mathcal{H}_c = \mathbf{D}_1\mathcal{H}_f(\mathbf{g}, \boldsymbol{\Gamma}(\mathbf{g}))$. See [4, Theorem 4.2] for the details.

For the sub-class of (75) where $\boldsymbol{\varphi}(\mathbf{g}) = (\varphi_1(g_1), \varphi_2(g_2), \varphi_3(g_3))$, so that the symmetry condition in Proposition 7 is satisfied, the Hamiltonian is quite easily obtained. We shall use this to demonstrate the application of the generalized Serret–Andoyer transformation for the rigid body with three rotors, bearing in mind that even in the general case, the same can be done if the expression of $\boldsymbol{\varphi}$ is given.

Corollary 2. *Let $\varphi_i : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$ functions, $i = 1, \dots, 3$. Then, the feedback control $\mathbf{u} = (u_1, u_2, u_3)(\mathbf{g})$ for the system* (74)*, defined by*

$$u_i = \varphi_i'(g_i)\dot{g}_i, \quad i \in \{1, 2, 3\} \tag{76}$$

preserves the rigid body structure in the sense of Lemma 7, with the closed-loop Hamiltonian

$$\mathcal{H}_c(\mathbf{g}) = \sum_{i=1}^{3} \frac{1}{\lambda_i} \int (g_i - \varphi_i(g_i) - G_i)\, dg_i, \tag{77}$$

where G_1, G_2 and G_3 are constants. Moreover, if $\varphi_i'(v) \neq 1$, $i = 1, \dots, 3$, then $\mathcal{H}_c$ is hyperregular.

Proof. Observe by expanding (74) that the control (76) conserves the quantities $G_i = \Gamma_i - \varphi(g_i)$, $i = 1, \dots, 3$. Then it can easily be verified by taking partial derivatives of (77) that (64) is satisfied. The inverse Legendre transform of the controlled Hamiltonian, $\boldsymbol{\omega} = \nabla_{\mathbf{g}} \mathcal{H}_c$, relates the body controlled momentum $\mathbf{g}$ and the body angular velocity $\boldsymbol{\omega}$ by

$$\omega_i = (g_i - \varphi_i(g_i) - G_i)/\lambda_i$$

for $i = 1, \dots, 3$. Since $\varphi_i'(v) \neq 1$ for all $v \in \mathbb{R}$, the above is invertible with differentiable inverse by the implicit function theorem. Hence, $\mathcal{H}_c$ is hyperregular.

The following are then immediate applications of Theorem 3.

Theorem 7. *Suppose $\varphi_i'(v) \neq 1$, $i = 1, \dots, 3$, for all $v \in \mathbb{R}$. Let $\tilde{\varphi}_1 = \varphi_1 \circ \mathcal{U}_1$ and $\tilde{\varphi}_2 = \varphi_2 \circ \mathcal{U}_2$. Then the closed-loop main body motion of the system* (74) *with the control* (76) *is reduced by the generalized Serret–Andoyer transformation to $\dot{l} = \partial \mathfrak{h}_{\mathbf{G}}/\partial L$, $\dot{L} = -\partial \mathfrak{h}_{\mathbf{G}}/\partial l$ with the reduced Hamiltonian*

$$\mathfrak{h}_{\mathbf{G}}(l, L) = \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) + \frac{L^2}{2\lambda_3} - \frac{1}{\lambda_3}\int (\varphi_3(L) + G_3) dL$$
$$- \sqrt{G^2 - L^2} \int \left[(\tilde{\varphi}_1(l, L) + G_1)\frac{\cos l}{\lambda_1} - (\tilde{\varphi}_2(l, L) + G_2)\frac{\sin l}{\lambda_2}\right] dl. \tag{78}$$

Moreover, in the spatial frame defined in Proposition 2, the closed-loop motion of the main body is described by the 3-1-3 Euler angles (ϕ, θ, l), with $\cos\theta = L/G$ and $\phi = \int \partial \mathcal{H}_c/\partial \Phi|_{\mathcal{M}_{\mathbf{G}}} dt$.

For the particular case where $\varphi_1 = \varphi_2 = 0$, that is, control is applied to rotor 3 only, the momenta Γ_1 and Γ_2 are integrals of motion. We thus have the following.

Corollary 3. *Under the conditions of Theorem 7, and choosing $\varphi_1 = \varphi_2 = 0$, the reduced Hamiltonian is then given by*

$$\mathfrak{h}_{\mathbf{G}}(l, L) = \frac{1}{2}(G^2 - L^2)\left(\frac{\sin^2 l}{\lambda_1} + \frac{\cos^2 l}{\lambda_2}\right) - \left(\frac{\Gamma_1 \sin l}{\lambda_1} + \frac{\Gamma_2 \cos l}{\lambda_2}\right)\sqrt{G^2 - L^2}$$
$$+ \frac{L^2}{2\lambda_3} - \frac{1}{\lambda_3}\int (\varphi_3(L) + G_3) dL, \tag{79}$$

where Γ_1, Γ_2 and G_3 are constants.

8. CONCLUSIONS

In this paper, we have completed reviewing the Serret–Andoyer (SA) formalism for modeling and control of rigid-body dynamics from the dynamical systems perspective. We have dwelt upon modeling control torques, geometry, and Stabilizing control. All these issues were embedded into the SA setup and new insights were contributed; in particular, we have shown that the SA formalism is very useful for global Stabilization of the rigid-body dynamics.

The SA formalism permits a straightforward derivation of asymptotically Stabilizing controllers and naturally supports practical problems such as satellite detumbling. However, the main feature of the Andoyer variables is the reduction of the underlying dynamics. The canonical approach facilitates the design of Stabilizing controllers, with the Hamiltonian serving as a natural Lyapunov function for the problem.

We have also shown that the symmetry-reduced phase space of the Andoyer variables naturally supports the synthesis of structure-preserving control through the mechanism of controlled Lagrangians.

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REFERENCES

1. Gurfil, P., Elipe, A., Tangren, W., and Efroimsky, M., The Serret—Andoyer Formalism in Rigid-Body Dynamics: I. Symmetries and Pertrubations, *Regul. Chaotic Dyn.*, 2007, vol. 12, no 4, pp. 389—425.
2. Abraham, R. and Marsden, J.E., *Foundations of Mechanics. Second Edition, Revised and Enlarged*, Reading, Mass.:Benjamin/Cummings Publishing Co., Inc., 1978.
3. Arnold, V.I. *Mathematical Methods of Classical Mechanics*, 2nd ed., V. 60 of *Graduate Texts in Mathematics*, New York: Springer-Verlag, 1989.
4. Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., and Sánchez De Alvarez, G., Stabilization of Rigid Body Dynamics by Internal and External Torques, *Automatica*, 1992, vol. 28, pp. 745—756.
5. Bloch, A.M., Leonard, N., and Marsden, J.E. Controlled Lagrangians and the Stabilization of Euler—Poincaré Mechanical Systems, *Internat. J. Robust Nonlinear Control*, 2001, vol. 11, pp. 191—214.
6. Bloch, A.M., *Nonholonomic Mechanics and Control*, vol. 24 of *Interdisciplinary Applied Mathematics. Systems and Control*, New York: Springer-Verlag, 2003.
7. Marsden, J.E. and Ratiu, T.S. *Introduction to Mechanics and Symmetry*, 2nd ed., New York: Springer-Verlag, 1999.
8. Kinoshita, H., Theory of the Rotation of the Rigid Earth, *Celestial Mech.*, 1977, vol. 15, pp. 277—326.
9. Laskar, J. and Robutel, J., The Chaotic Obliquity of the Planets, *Nature*, 1993, vol. 361, pp. 608—612.
10. Touma, J. and Wisdom, J., Lie—Poisson Integrators for Rigid Body Dynamics in the Solar System, *The Astronomical Journal*, 1994, vol. 107, pp. 1189—1202.
11. Efroimsky, M. and Escapa, A., The Theory of Canonical Perturbations Applied to Attitude Dynamics and to the Earth Rotation. Osculating and Non-Osculating Andoyer Variables, *Celestial Mech. Dynam. Astronom.*, 2007, vol. 98, no 4, pp. 251—283.
12. Kinoshita, H., First-Order Perturbations of the Two Finite-Body Problem, *Publications of the Astronomical Society of Japan*, 1972, vol. 24, pp. 423—457.
13. Kane, T.R., Likins, P.W., and Levinson, D.A., *Spacecraft Dynamics*, New York: McGraw Hill, 1983.
14. Junkins, J.L. and Turner, J.D. *Optimal Spacecraft Rotational Maneuvers*, New York: Elsevier, 1986.
15. Schaub, H. and Junkins, J.L., *Analytical Mechanics of Space Systems*, AIAA Educational Series, Reston VA: AIAA, 2003.
16. Zanardi, M.C., Santos, R.M.K., and da Silva Fernandes, S. Optimal Attitude Corrections for Cylindrical Spacecraft, *Advances in Space Research*, 2003, vol. 31, no. 8, pp. 1987—1903.
17. Giacaglia, G.E.O. and Jefferys, W.H., Motion of a Space Station. I, *Celestial Mech.*, 1971, vol. 4, pp. 442—467.
18. Bloch, A.M., and Marsden, J.E., Stabilization of Rigid Body Dynamics by the Energy Casimir Method, Systems Control Lett., 1990, vol. 15, pp. 341—346.
19. Bullo, F., Nonlinear Control of Mechanical Systems: A Riemannian Geometry Approach, Ph.D. thesis, California Institute of Technology, 1998.
20. Tsiotras, P. Further Passivity Results for the Attitude Control Problem, *IEEE Trans. Automat. Control*, 1998, vol. 43, no. 11, pp. 597—1600.
21. Ioslovich, I. Optimal Control of Rigid Body Rotation Around Center of Mass, *J. Dynam. Control Systems*, 2003, vol. 9, pp. 549—562.
22. Andriano, V., Global Feedback Stabilization of the Angular Velocity of a Symmetric Rigid Body, *Systems Control Lett.*, 1993, vol. 20, no. 5, pp. 361—364.
23. Aeyels, D., Stabilization by Smooth Feedback of the Angular Velocity of a Rigid Body, *Systems Control Lett.*, 1985, vol. 6, no. 1, pp. 59—63.
24. Astolfi, A., Output Feedback Control of the Angular Velocity of a Rigid Body, *Systems Control Lett.*, 1999, vol. 36, no. 3, pp. 181—192.
25. Astolfi, A. and Rapaport, A., Robust Stabilization of the Angular Velocity of a Rigid Body, *Systems Control Lett.*, 1998, vol. 34, no. 5, pp. 257—264.

26. Paynter, S. J. and Bishop R.H., Adaptive Nonlinear Attitude Control and Momentum Management of Spacecraft, *Journal of Guidance, Control and Dynamics*, 1997, vol. 20, no. 5, pp. 1025–1032.
27. Lo, S.-C. and Chen, Y.-P., Smooth Sliding-Mode Control for Spacecraft Attitude Tracking Maneuvers, *Journal of Guidance, Control, and Dynamics*, 1995, vol. 18, pp. 1345–1349.
28. Shuster, M. 1993. A Survey of Attitude Representations, *J. Astronaut. Sci.*, vol. 41, pp. 439–517.
29. Lum, K.-Y. and Bloch, A.M., Generalized Serret–Andoyer Transformation and Applications for the Controlled Rigid Body, *Dynam. Control*, 1999, vol. 9, pp. 39–66.
30. Bryson, A.E. and Ho, Y., *Applied Optimal Control.*, Washington DC: Hemisphere Pub- lishing Co., 1975.
31. Jurdjevic, V. and Quinn, J.P., Controllability and Stability, *J. Differential Equations*, 1978, vol. 28, pp. 381–389.
32. Sontag, E.D., *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed., New York: Springer-Verlag, 1998.
33. Clarke, F.H., Ledyaev, Y.S., Subbotin, A.I., and Sontag, E. D., Asymptotic Controllability implies Feedback Stabilization, *IEEE Trans. Automat. Control*, 1997, vol. 42, pp. 1394–1407.
34. Sepulchre, R., Janković, M., and Kokotović, P., Constructive Nonlinear Control, Berlin: Springer-Verlag, 1997.
35. Tsiotras, P., Stabilization and Optimality Results for the Attitude Control Problem, *Journal of Guidance, Control, and Dynamics*, vol. 19, no. 4, 1996, pp. 772–779.
36. Primbs, J.A., Nevistić, V., and Doyle, J.C., Nonlinear Optimal Control: A Control Lyapunov Function and Receding Horizon Perspective, *Asian Journal of Control*, 1999, vol. 1, no. 1, pp. 14–24.