ANALYSIS OF J_2-PERTURBED MOTION USING MEAN NON-OSCULATING ORBITAL ELEMENTS

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Abstract. This paper investigates the long-period and secular dynamics of a satellite about an oblate primary while relieving the assumption that the perturbed orbit is instantaneously parameterized by osculating Keplerian orbits. The inherent freedom obtained by transforming the orbital dynamics from the Cartesian inertial space to the orbital elements space, termed gauge freedom, is utilized to nullify four planetary equations. It is shown that there exists an orbit representation in which the mean non-osculating perigee is stable under the oblateness perturbation and that nodal precession, apsidal rotation and epoch drift may be simultaneously nullified on the expense of secular eccentricity and inclination variations. These observations considerably expand the standard description of J_2-perturbed motion using mean osculating orbital elements, which predicts secular variation nullification of semi-major axis, eccentricity and inclination only.

Key words: oblateness, Lagrange's equation, orbital perturbations, orbital elements, gauge invariance

1. Introduction

Lagrange's planetary equations (Euler, 1748; Lagrange, 1809) (LPEs) have been used in celestial mechanics and astrodynamics for well over two centuries. These time-dependent, ordinary differential equations describe the variation of the classical orbital elements due to a disturbing potential input. From the mathematical standpoint, these equations map the orbital dynamics from the position-velocity space to the orbital elements space through variations-of-parameters (VOP), a general and powerful technique for solving non-linear differential equations, developed by Euler (1748) and later enhanced by Lagrange (1809).

What can be contributed to this long-standing problem? The mathematical development leading to the re-formulation of the problem dynamics in terms of orbital elements gives rise to an underdetermined system, meaning that extra constrains should be imposed to solve for the excess freedom.

To eliminate this freedom, Lagrange and most of his followers assumed that the velocity vector of the perturbed orbit equals the velocity vector of the generating Keplerian orbit, thus imposing three additional constraints known as the Lagrange constraints. The orbital elements resulting from this assumption are known as osculating orbital elements (Battin, 1999), because the trajectory in the inertial configuration space is always tangential to an 'instantaneous' ellipse (or hyperbola) defined by the 'instantaneous' values of the time-varying orbital elements. This means that the perturbed physical trajectory would coincide with the Keplerian orbit that the body would follow if the perturbing force was to cease instantaneously.

While the Lagrange constraint greatly simplifies the analysis, it avoids tackling the hidden symmetry of the equations. The generalized constraint may be utilized as a user-defined tuning mechanism for the planetary equations, leading to possible simplification of numerical integration and to a better understanding of the underlying problem dynamics.

This observation has been made by a few researchers (Brouwer and Clemence, 1961). Recently, there has been a marked increase in investigation of generalizations to the Lagrange constraint as Efroimsky et al. have published a number of key works in this area (Efroimsky, 2002; Efroimsky and Goldreich, 2003, in press; Newman and Efroimsky, 2003). The resulting planetary equations were termed *gauge-generalized* equations, and the underlying symmetry was referred to as *gauge symmetry* or *gauge freedom*, a terminology adopted from the field of electrodynamics.

In this paper, we extend the existing results by presenting an averaged form of the gauge-generalized planetary equations. This representation is beneficial for studying secular and long-period effects on the orbital dynamics due to a first-order small perturbing potential. The resulting equations yield gauge-generalized planetary equations with mean non-osculating classical orbital elements as the state variables. We then utilize the equations in order to develop a model of satellite motion about an oblate planet, taking into account only the first zonal harmonics (J_2).

J_2-perturbed motion has been thoroughly studied in the existing literature, both from the artificial and natural satellite standpoints. A myriad of works have been published on analytic methods (Palmer and Hashida, 2001), oblateness-perturbed relative motion (Koon and Murray, 2001; Schaub and Alfriend, 2001), numerical integration (Hadjifotinou, 2000) and even utilization of oblateness for constraining the rotation rate of extra-solar planets (Seager and Hui, 2002). However, the bulk of the works thus far have utilized osculating orbital elements. The models herein are developed using *non-osculating* elements. The excess freedom introduced by using the gauge-generalized equations is used to identically nullify three planetary equations in addition to the semi-major axis rate. We have thus managed to nullify four planetary equations whereas in the osculating case only three orbital element rates are nullified (semi-major axis, eccentricity and inclination). We illustrate the methodology by presenting oblateness-perturbed planetary equations with a stable perigee as well as planetary equations in which the nodal precession, apsidal rotation and drift of the epoch are cancelled on the expense of eccentricity and inclination variation.

2. Background

Consider the equations of motion of a Keplerian two-body problem, given by

$$\ddot{\mathbf{r}} + \frac{\mu \mathbf{r}}{r^3} = 0, \tag{1}$$

where $\mathbf{r} \in \mathbb{R}_0^3$ is the position vector in some inertial coordinate system, $r = \|\mathbf{r}\|$, $\mathbb{R}_0^3 \triangleq \mathbb{R}^3 \backslash \{0\}$ is the Cartesian configuration space and μ is the gravitational constant. In order to solve Equation (1), we first write the position vector in an auxiliary dextral perifocal coordinate system, $\mathscr{P}$:

$$\mathbf{r}_{\mathscr{P}} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix}, \tag{2}$$

where r is given by the conic equation

$$r = \frac{a(1-e^2)}{1+e\cos f}, \tag{3}$$

a is the semi-major axis, e is the eccentricity, $f = f(a, e, M_0, t)$ is the true anomaly and M_0 is the mean anomaly at epoch. The perifocal velocity vector is obtained by writing

$$\dot{\mathbf{r}}_{\mathscr{P}} = \dot{f}\left[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}f}\right]_{\mathscr{P}} = \sqrt{\frac{\mu}{a^3(1-e^2)^3}}(1+e\cos f)^2\left[\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}f}\right]_{\mathscr{P}}, \tag{4}$$

which yields

$$\dot{\mathbf{r}}_{\mathscr{P}} = \begin{bmatrix} -\sqrt{\frac{\mu}{a(1-e^2)}}\sin f \\ \sqrt{\frac{\mu}{a(1-e^2)}}(e+\cos f) \\ 0 \end{bmatrix}. \tag{5}$$

In order to obtain $\mathbf{r}$, the inertial position vector, and $\dot{\mathbf{r}}$, the inertial velocity vector, we need to use a rotation matrix from perifocal to inertial coordinates, $T_{\mathscr{P}}^{\mathscr{I}} \in SO(3)$. One such transformation is given by (Bate and White, 1971)

$$T_{\mathscr{P}}^{\mathscr{I}}(i,\Omega,\omega) = \begin{bmatrix} \mathrm{c}(\Omega)\mathrm{c}(\omega) - \mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i) & -\mathrm{c}(\Omega)\mathrm{s}(\omega) - \mathrm{s}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i) & \mathrm{s}(\Omega)\mathrm{s}(i) \\ \mathrm{s}(\Omega)\mathrm{c}(\omega) + \mathrm{c}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i) & -\mathrm{s}(\Omega)\mathrm{s}(\omega) + \mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i) & -\mathrm{c}(\Omega)\mathrm{s}(i) \\ \mathrm{s}(\omega)\mathrm{s}(i) & \mathrm{c}(\omega)\mathrm{s}(i) & \mathrm{c}(i) \end{bmatrix}, \tag{6}$$

where i is the inclination, Ω is the longitude of the ascending node, ω is the argument of periapsis and we used the compact notation $\mathrm{c}(\cdot) = \cos(\cdot)$, $\mathrm{s}(\cdot) = \sin(\cdot)$. Transforming into inertial coordinates, using Equations (2) and (6), we obtain the general solution to Equation (1),

$$\mathbf{r} = T_{\mathscr{P}}^{\mathscr{I}}(\Omega, \omega, i)\mathbf{r}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{f}(a, e, i, \Omega, \omega, M_0, t), \tag{7}$$

where M_0 is the mean anomaly at epoch. In a similar fashion, the expression for the inertial velocity is given by

$$\dot{\mathbf{r}} = T_{\mathscr{P}}^{\mathscr{I}}(\Omega, w, i)\dot{\mathbf{r}}_{\mathscr{P}}(a, e, M_0, t) = \mathbf{g}(a, e, i, \Omega, \omega, M_0, t). \tag{8}$$

Thus, the inertial position and velocity depend upon time and the classical orbital elements,

$$\boldsymbol{\alpha} = [a, e, i, \Omega, \omega, M_0]^{\mathrm{T}} \in \mathcal{M}, \tag{9}$$

where $\mathcal{M} = \mathcal{O} \times \mathbb{S}^4$, $\mathcal{O} \subset \mathbb{R}^2$ is an open set in $\mathbb{R}^2$, and $\mathbb{S}^4$ is the 4-sphere.

The above solutions were obtained for the nominal, undisturbed Keplerian motion. When a disturbing specific force, $\mathbf{d}$, is introduced into Equation (1), we have

$$\ddot{\mathbf{r}} + \frac{\mu\mathbf{r}}{r^3} = \mathbf{d}. \tag{10}$$

In order to solve for the resulting non-Keplerian motion, Euler (1999) and Lagrange (2002) have developed the variation-of-parameters procedure (which is a general and powerful method for the solution of non-linear differential equations). In essence, the method suggests to turn the constants of the unperturbed motion, which in our case are the classical orbital elements, into functions of time, yielding a modified solution of the form

$$\mathbf{r} = \mathbf{f}(\boldsymbol{\alpha}(t), t). \tag{11}$$

Taking the time derivative of Equation (11) yields the relationship

$$\dot{\mathbf{r}} = \frac{\partial \mathbf{f}}{\partial t} + J_{\mathbf{f}}(\boldsymbol{\alpha}, t)\dot{\boldsymbol{\alpha}} = \mathbf{g}(\boldsymbol{\alpha}(t), t) + J_{\mathbf{f}}(\boldsymbol{\alpha}, t)\dot{\boldsymbol{\alpha}}, \tag{12}$$

where $J_{\mathbf{f}}(\boldsymbol{\alpha}, t) = \partial\mathbf{f}/\partial\boldsymbol{\alpha}$. In order to obtain the differential equations describing the temporal change of the classical orbital elements, known as Lagrange's planetary equations (LPEs), Equation (12) is differentiated and substituted into Equation (10). This operation results in a 12-dimensional system of differential equations for $\boldsymbol{\alpha}$ and $\dot{\boldsymbol{\alpha}}$. However, there are only three degrees of freedom. Hence, the resulting system will be under-determined, meaning that three extra conditions can be imposed. Lagrange chose to impose the non-holonomic constraint

$$J_{\mathbf{f}}(\boldsymbol{\alpha}, t)\dot{\boldsymbol{\alpha}} = \mathbf{0}, \tag{13}$$

which is also known as the *Lagrange constraint*. Mathematically, this restriction confines the dynamics of the orbital state space to a 9-dimensional submanifold of the 12-dimensional manifold $\mathcal{M} \times \mathbb{R}^6$. More importantly, this freedom reflects an internal symmetry in the mapping $(\mathbf{r}, \dot{\mathbf{r}}) \mapsto (\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}})$, which is inherent to Lagrange's planetary equations.

Physically, the Lagrange constraint postulates that the trajectory in the inertial configuration space is always tangential to an 'instantaneous' ellipse (or hyperbola) defined by the 'instantaneous' values of the time-varying orbital elements $\boldsymbol{\alpha}(t)$, meaning that the perturbed physical trajectory would coincide with the Keplerian orbit that the body would follow if the perturbing force was to cease instantaneously. This instantaneous orbit is called *osculating orbi*. Accordingly, the orbital elements which, satisfy the Lagrange constraint are called *osculating orbital elements*.

Although the Lagrange constraint simplifies the calculations and can be interpreted using Keplerian dynamics, it is *completely arbitrary*. The generalized form of the Lagrange constraint may be therefore written as

$$J_{\mathbf{F}}(\boldsymbol{\alpha}, t)\dot{\boldsymbol{\alpha}} = \mathbf{q}(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}, t), \tag{14}$$

where $\mathbf{q} : \mathcal{M} \times \mathbb{R}^6 \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ is an *arbitrary* function of the classical orbital elements, their-time derivatives and time. This important observation has been made by a few researchers (Brouwer and Clemence, 1961). Recently, Efroimsky et al. have published important works on planetary equations with a generalized Lagrange constraint (Efroimsky, 2002; Efroimsky and Goldreich, 2003, in press; Newman and Efroimsky, 2003). They termed the underlying symmetry *gauge symmetry* and the constraint function $\mathbf{q}$ *gauge function*, which are terms taken from the field of electrodynamics. The gauge $\mathbf{q} = 0$ is termed the *Lagrange gauge*.

The use of a generalized Lagrange constraint gives rise to *non-osculating* orbital elements, which relate to the inertial position and velocity based upon Equations (11) and (12), respectively. Thus, although the description of the physical orbit in the inertial Cartesian configuration space remains invariant to a particular selection of a gauge function, its description in the orbital elements space depends on whether osculating or non-osculating orbital elements are used.

This fact is illustrated by Figure 1, which depicts a perturbed Keplerian orbit in an inertial reference frame $\widehat{X}\widehat{Y}\widehat{Z}$ and an instantaneous velocity vector $\mathbf{v}$. The velocity vector defines an osculating ellipse at the point P. Yet, point P may lie on another, non-osculating ellipse. We shall show in the forthcoming sections that the use of non-osculating elements is beneficial for studying the dynamics of J_2-perturbed orbits.

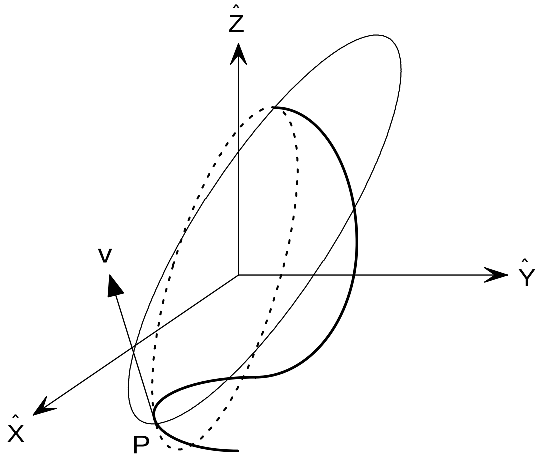

Figure 1. A perturbed Keplerian orbit (thick curve). The position at P can be described by an osculating ellipse, tangent to the instantaneous velocity vector $\mathbf{v}$ (dashed), or a non-osculating ellipse (thin line).

3. Gauge-Generalized Averaged Planetary Equations

When osculating orbital elements are used, the LPEs may be compactly written as

$$\dot{\boldsymbol{\alpha}} = [J_{\mathbf{f}}(\boldsymbol{\alpha})P]^{\mathrm{T}}\mathbf{d} \tag{15}$$

where P is the 6×6 skew-symmetric *Poisson matrix* (Battin, 1999), given by

$$P^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{2}{na} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{1-e^2}}{na^2 e} & \frac{1-e^2}{na^2 e} \\ 0 & 0 & 0 & -\frac{1}{na^2\sqrt{1-e^2}\sin i} & \frac{\cot i}{na^2\sqrt{1-e^2}} & 0 \\ 0 & 0 & \frac{1}{na^2\sqrt{1-e^2}\sin i} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{1-e^2}}{na^2 e} & -\frac{\cot i}{na^2\sqrt{1-e^2}} & 0 & 0 & 0 \\ -\frac{2}{na} & -\frac{1-e^2}{na^2 e} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

Assuming that the disturbing specific force is conservative and depends on the position vector only, we may substitute $\mathbf{d}$ in Equation (15) by the gradient of a disturbing potential, denoted by R, such that $\mathbf{d} = \partial R/\partial \mathbf{r}$. Since

$$\nabla R = \frac{\partial R}{\partial \boldsymbol{\alpha}} = J_{\mathbf{f}}^{\mathrm{T}}(\boldsymbol{\alpha})\frac{\partial R}{\partial \mathbf{r}}, \tag{17}$$

the LPEs in osculating orbital elements become

$$\dot{\boldsymbol{\alpha}} = P^{\mathrm{T}}\nabla R. \tag{18}$$

A similar procedure can be carried out in order to derive the LPEs in non-osculating orbital elements using the gauge-generalized constraint (14). Efroimsky et al. (Efroimsky, 2002; Newman and Efroimsky, 2003) have shown that in this case the LPEs are

$$\dot{\boldsymbol{\alpha}} = P^{\mathrm{T}}[\nabla R - (J_{\mathbf{g}}^{\mathrm{T}}\mathbf{q} + J_{\mathbf{f}}^{\mathrm{T}}\dot{\mathbf{q}})], \tag{19}$$

where $J_{\mathbf{g}} = \partial\mathbf{g}/\partial\boldsymbol{\alpha}$, $\dot{\mathbf{q}} = J_{\mathbf{q}}\dot{\boldsymbol{\alpha}}$ and $J_{\mathbf{q}} = \partial\mathbf{q}/\partial\boldsymbol{\alpha}$.

We shall now assume that the gauge function $\mathbf{q}$ is a function of orbital elements *only*,

$$\mathbf{q} : \mathcal{M} \to \mathbb{R}^3 = \mathbf{q}(\boldsymbol{\alpha}), \tag{20}$$

and derive the averaged form of Equation (19) using the method of Brouwer and Clemence (1961). Essentially, the averaging procedure yields expressions for the secular effect of first-order perturbations on an orbit assuming that the variations of orbital elements during the averaging is of second order. The resulting orbital elements are called *mean* orbital elements.

The averaging of the perturbing potential is carried out as follows (Battin, 1999):

$$\bar{R} = \langle R \rangle = \frac{1}{2\pi}\int_0^{2\pi} R\,\mathrm{d}M \tag{21}$$

or, alternatively,

$$\bar{R} = \frac{1}{2\pi}\int_0^{2\pi} \frac{n}{h} R r^2\,\mathrm{d}f, \tag{22}$$

where $n = \sqrt{\mu/a^3}$ is the mean motion and $h = \sqrt{\mu a(1-e^2)}$ is the orbital angular momentum. The mean value of some vector $\mathbf{p} = [p_1, p_2, p_3]^{\mathrm{T}}$ is obtained by performing the averaging (22) component-wise, $\langle \mathbf{p} \rangle = [\bar{p}_1, \bar{p}_2, \bar{p}_3]^{\mathrm{T}}$,

To perform the averaging procedure, we assume that the perturbing potential is first-order small, that is,

$$R = \varepsilon\widetilde{R}, \quad \varepsilon \ll 1, \tag{23}$$

and in addition, that the (arbitrary) gauge function satisfies

$$\mathbf{q} = \varepsilon\tilde{\mathbf{q}}, \quad \varepsilon \ll 1. \tag{24}$$

These assumptions guarantee that to first order, the mean orbital elements remain unchanged in the interval $f = [0, 2\pi]$. Hence, the averaged differential equations for the mean orbital elements are obtained by substituting

$$\boldsymbol{\alpha} = \bar{\boldsymbol{\alpha}}, \quad \dot{\boldsymbol{\alpha}} = \dot{\bar{\boldsymbol{\alpha}}} \tag{25}$$

into the planetary equations. If we now substitute Equation (25) into Equation (19) and apply the averaging (22), we shall obtain differential equations for the mean non-osculating orbital elements:

$$\dot{\bar{\boldsymbol{\alpha}}} = P^{\mathrm{T}}(\bar{\boldsymbol{\alpha}})[\nabla \bar{R}(\bar{\boldsymbol{\alpha}}) - \langle J_{\mathbf{g}}^{\mathrm{T}}\mathbf{q}\rangle - \langle J_{\mathbf{f}}^{\mathrm{T}}\dot{\mathbf{q}}\rangle]. \tag{26}$$

Based on the relationship (25), the interchangeability of the integration and differentiation operators and assumption (20), we obtain the equalities

$$\langle J_{\mathbf{g}}^{\mathrm{T}}\mathbf{q}\rangle = \langle J_{\mathbf{g}}^{\mathrm{T}}\rangle\langle\mathbf{q}\rangle = \frac{\partial\langle T_{\mathscr{P}}^{\mathscr{I}}\dot{\mathbf{r}}_{\mathscr{P}}\rangle}{\partial\bar{\boldsymbol{\alpha}}}\bar{\mathbf{q}}. \tag{27}$$

Substituting Equations (5) and (6) into Equation (27) and performing the averaging yields

$$\langle T_{\mathscr{P}}^{\mathscr{I}}\dot{\mathbf{r}}_{\mathscr{P}}\rangle = \dot{\bar{\mathbf{r}}} = \mathbf{0}. \tag{28}$$

In a similar manner,

$$\langle J_{\mathbf{f}}^{\mathrm{T}}\dot{\mathbf{q}}\rangle = \langle J_{\mathbf{f}}^{\mathrm{T}}\rangle\langle\dot{\mathbf{q}}\rangle = \frac{\partial\langle T_{\mathscr{P}}^{\mathscr{I}}\mathbf{r}_{\mathscr{P}}\rangle}{\partial\bar{\boldsymbol{\alpha}}}\langle\dot{\mathbf{q}}\rangle, \tag{29}$$

where

$$\langle T_{\mathscr{P}}^{\mathscr{I}}\mathbf{r}_{\mathscr{P}}\rangle = \bar{\mathbf{r}} = \frac{3}{2}ae\begin{bmatrix} \sin\Omega\sin\omega\cos i - \cos\Omega\cos\omega \\ -\cos\Omega\sin\omega\cos i - \sin\Omega\cos\omega \\ -\sin i\sin\omega \end{bmatrix} \tag{30}$$

and the elements of $J_{\mathbf{f}}(\bar{\boldsymbol{\alpha}}) = \partial\bar{\mathbf{r}}/\partial\bar{\boldsymbol{\alpha}}$ are given by

$$\begin{aligned}
&(\bar{J}_{\mathbf{f}})_{1,1} = e[\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i) - \mathrm{c}(\Omega)\mathrm{c}(\omega)], \quad (\bar{J}_{\mathbf{f}})_{1,2} = a[\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i) - \mathrm{c}(\Omega)\mathrm{c}(\omega)],\\
&(\bar{J}_{\mathbf{f}})_{1,3} = -ae\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}, \quad (\bar{J}_{\mathbf{f}})_{1,4} = ae[\mathrm{s}(\Omega)\mathrm{c}(\omega) + \mathrm{c}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)](i),\\
&(\bar{J}_{\mathbf{f}})_{1,5} = ae[\mathrm{c}(\Omega)\mathrm{s}(\omega) + \mathrm{s}(\Omega)\mathrm{c}(\omega)c(i)],\\
&(\bar{J}_{\mathbf{f}})_{2,1} = -e[\mathrm{s}(\Omega)\mathrm{c}(\omega) + \mathrm{c}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)],\\
&(\bar{J}_{\mathbf{f}})_{2,2} = -a[\mathrm{s}(\Omega)\mathrm{c}(\omega) + \mathrm{c}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)],\\
&(\bar{J}_{\mathbf{f}})_{2,3} = ae[\mathrm{c}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)], \quad (\bar{J}_{\mathbf{f}})_{2,4} = -ae[\mathrm{c}(\Omega)\mathrm{c}(\omega) - \mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)],\\
&(\bar{J}_{\mathbf{f}})_{2,5} = -\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)], -\mathrm{s}(\Omega)\mathrm{s}(\omega), \quad (\bar{J}_{\mathbf{f}})_{3,1} = -e\mathrm{s}(\omega)\mathrm{s}(i),\\
&(\bar{J}_{\mathbf{f}})_{3,2} = -a\mathrm{s}(\omega)\mathrm{s}(i), \quad (\bar{J}_{\mathbf{f}})_{3,3} = -ae\mathrm{s}(\omega)\mathrm{c}(i),\\
&(\bar{J}_{\mathbf{f}})_{3,5} = -\mathrm{c}(\omega)\mathrm{s}(i), \quad (\bar{J}_{\mathbf{f}})_{1,6} = (\bar{J}_{\mathbf{f}})_{2,6} = (\bar{J}_{\mathbf{f}})_{3,6} = (\bar{J}_{\mathbf{f}})_{3,4} = 0.
\end{aligned} \tag{31}$$

The calculation of $\langle\dot{\mathbf{q}}\rangle$ is a bit more subtle, due to the fact that $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}, f)$. To begin, we note that based on assumption (20), $\partial\mathbf{q}/\partial t = \mathbf{0}$, so that

$$\dot{\mathbf{q}} = J_{\mathbf{q}}(\boldsymbol{\alpha})\frac{\partial\boldsymbol{\alpha}}{\partial t} = J_{\mathbf{q}}(\boldsymbol{\alpha})\frac{\partial\boldsymbol{\alpha}}{\partial f}\dot{f} = J_{\mathbf{q}}(\boldsymbol{\alpha})\frac{\partial\boldsymbol{\alpha}}{\partial f}\frac{h}{r^2}, \tag{32}$$

where $J_{\mathbf{q}}(\boldsymbol{\alpha}) = \partial\mathbf{q}/\partial\boldsymbol{\alpha}$. Substituting into (22) we obtain

$$\langle\dot{\mathbf{q}}\rangle = J_{\mathbf{q}}(\boldsymbol{\alpha})\frac{n}{2\pi}\int_0^{2\pi}\frac{\partial\boldsymbol{\alpha}}{\partial f}\,\mathrm{d}f = \mathbf{Q}(\boldsymbol{\alpha}) = \mathbf{Q}(\bar{\boldsymbol{\alpha}}). \tag{33}$$

Substituting Equations (28) and (33) into Equation (26) we obtain a simplified set of gauge-generalized, averaged, planetary equations:

$$\dot{\bar{\boldsymbol{\alpha}}} = P^{\mathrm{T}}(\bar{\boldsymbol{\alpha}})[\nabla\bar{R}(\bar{\boldsymbol{\alpha}}) - J_{\mathbf{f}}^{\mathrm{T}}(\bar{\boldsymbol{\alpha}})\mathbf{Q}(\bar{\boldsymbol{\alpha}})], \tag{34}$$

where $J_{\mathbf{f}}(\bar{\boldsymbol{\alpha}})$ is given by Equation (31).

We may utilize Equation (34) in order to find a vector function $\mathbf{Q}(\bar{\boldsymbol{\alpha}})$ that will *identically* nullify (at most) three averaged planetary equations for *almost all* perturbing potentials. This is an important observation, as it may significantly simplify orbit integration procedures (e.g. replace Cowell's method) and give new insight into the perturbed orbital dynamics. In the next section, we shall derive gauge-invariant equations for a J_2-perturbed orbit using mean non-osculating orbital elements, and illustrate how to find a gauge function that identically nullify various combinations of three planetary equations.

Alternatively, one may wish to find the least-squares solution to Equation (26), given by

$$\mathbf{Q} = (J_{\mathbf{f}}^{\mathrm{T}})^{+}\,\nabla\bar{R}, \tag{35}$$

where $(\cdot)^{+}$ denotes the Moore-Penrose generalized inverse of a rectangular matrix. The meaning of such a solution would be a 'minimum-dirt' parametrization of a perturbed orbit in the least-squares sense.

4. Mean Non-Osculating J_2 Equations

The mean perturbing potential due to an oblate primary is obtained by expanding the perturbing potential into a Fourier series in the mean anomaly M and averaging the first term. This procedure yields (Battin, 1999):

$$\bar{R} = \frac{\mu J_2 r_{\mathrm{eq}}^2(2 - 3\sin^2 i)}{4(1-e^2)^{\frac{3}{2}}}, \tag{36}$$

where r_{eq} is the equatorial radius. Computing the gradient with respect to the mean orbital elements yields

$$\nabla\bar{R} = \frac{3\mu J_2 r_{\mathrm{eq}}^2}{4a^3(1-e^2)^{\frac{3}{2}}} \begin{bmatrix} -\dfrac{(2-3\sin^2 i)}{a} \\ \dfrac{e(2-3\sin^2 i)}{(1-e^2)} \\ -\sin 2i \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{37}$$

We note that since $\partial\bar{R}/\partial M_0 = 0$ and $\partial\bar{\mathbf{f}}/\partial M_0 = 0$, the mean value of the semi-major axis remains unchanged regardless of any particular selection of a gauge function:

$$\dot{\bar{a}} = 0. \tag{38}$$

Selecting the Lagrange gauge rate $\mathbf{Q} = \mathbf{0}$ shows that the mean values of the osculating eccentricity and inclination are also invariant under the action of an oblateness perturbation. However, $\mathbf{Q} = \mathbf{0}$ is merely a special solution to the set of equations $\dot{\bar{e}} = 0$, $\dot{\bar{i}} = 0$. The gauge-generalized solution is obtained by substituting Equations (31) and (37) into Equation (34) and solving the resulting equations

$$\dot{\bar{e}} = -\frac{1-e^2}{na^2e}\left[\frac{\partial\bar{\mathbf{f}}}{\partial M_0}\right]^{\mathrm{T}}\mathbf{Q} + \frac{\sqrt{1-e^2}}{na^2e}\left[\frac{\partial\bar{\mathbf{f}}}{\partial\omega}\right]^{\mathrm{T}}\mathbf{Q} = 0, \tag{39}$$

$$\dot{\bar{i}} = -\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\left[\frac{\partial\bar{\mathbf{f}}}{\partial\omega}\right]^{\mathrm{T}}\mathbf{Q} + \frac{1}{na^2\sqrt{1-e^2}\sin i}\left[\frac{\partial\bar{\mathbf{f}}}{\partial\Omega}\right]^{\mathrm{T}}\mathbf{Q} = 0, \tag{40}$$

for $\mathbf{Q}$. Letting $\mathbf{Q} = [Q_1, Q_2, Q_3]^{\mathrm{T}}$, the general solution to the under-determined set of Equations (39) and (40) is obtained as

$$\mathbf{Q} = \begin{bmatrix} -\dfrac{Q_3(\cos i\sin\Omega\tan\omega - \cos\Omega)}{\tan\omega\sin i} \\ \dfrac{Q_3(\cos i\cos\Omega\tan\omega + \sin\Omega)}{\tan\omega\sin i} \\ Q_3 \end{bmatrix}. \tag{41}$$

The singularity at $\omega = 0, \pi, 2\pi$ and $i = 0, \pi$ is removable by selecting a differentiable Q_3 satisfying $Q_3 = Q_3(\omega, i)$ and $Q_3(\omega = 0, i = 0) = 0$, so that L'Hospital's rule may be applied (e.g. $Q_3 = c\sin\omega\sin i$ with $c = \text{const.}$)

The redundant degree of freedom may be now used to nullify an *additional* planetary equation. However, we must first verify that the resulting set of equations is solvable for $\mathbf{Q}$. Generally speaking, one may try and nullify any of the 10 combinations resulting from selecting any 3 out of the 5 remaining planetary equations. However, not all combinations are solvable for **Q**. The following four combinations do not admit a solution:

$$\begin{aligned}
&\dot{\bar{e}} = 0, \quad \dot{\bar{i}} = 0, \quad \dot{\bar{\Omega}} = 0,\\
&\dot{\bar{i}} = 0, \quad \dot{\bar{\Omega}} = 0, \quad \dot{\bar{M}}_0 = 0,\\
&\dot{\bar{i}} = 0, \quad \dot{\bar{\Omega}} = 0, \quad \dot{\bar{\omega}} = 0,\\
&\dot{\bar{e}} = 0, \quad \dot{\bar{\omega}} = 0, \quad \dot{\bar{M}} = 0.
\end{aligned} \tag{42}$$

The insolvability of the above combinations stems from the fact that the matrix of coefficients of the resulting linear system of equations is singular. The solution set is therefore the empty set.

The subsequent discussion illustrates a few solutions to some admissible combinations of planetary equations.

4.1. NULLIFYING THE APSIDAL LINE PRECESSION

The additional degree of freedom emerging from solution of Equations (39) and (40) may be used, e.g., to nullify the mean rate of change of the argument of perigee by solving the additional equation

$$\begin{aligned}
\dot{\bar{\omega}} = &-\frac{\cos i}{na^2\sqrt{1-e^2}\sin i}\left\{-3\sin 2i - \left[\frac{\partial\bar{\mathbf{f}}}{\partial i}\right]^{\mathrm{T}}\mathbf{Q}\right\}\\
&+\frac{\sqrt{1-e^2}}{na^2e}\left\{\frac{3e(2-3\sin^2 i)}{(1-e^2)} - \left[\frac{\partial\bar{\mathbf{f}}}{\partial e}\right]^{\mathrm{T}}\mathbf{Q}\right\} = 0.
\end{aligned} \tag{43}$$

Solving Equations (39), (40) and (43) for **Q** gives

$$\begin{aligned}
\mathbf{Q} = &\frac{\mu J_2 r_{\mathrm{eq}}^2 e[1-5\mathrm{c}^2(i)]}{4a^4(1-e^2)^{5/2}}\\
&\times\begin{bmatrix} 2[\mathrm{c}(\Omega)\mathrm{c}(\omega) - \mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)]\\ \dfrac{\mathrm{s}(2\omega)\mathrm{c}(i)[1-2\mathrm{c}^2(\Omega)] + \mathrm{s}(2\Omega)[\mathrm{c}^2(\omega)\mathrm{c}^2(i) + \mathrm{c}^2(i) + 1}{\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i) - \mathrm{c}(\Omega)\mathrm{c}(\omega)}\\ 2\mathrm{s}(\omega)\mathrm{s}(i)\end{bmatrix}.
\end{aligned} \tag{44}$$

This gauge function rate is regular, integrable $\forall\bar{\boldsymbol{\alpha}} \in \mathcal{M}\backslash\{i,\Omega,\omega : \mathrm{c}(i)\mathrm{s}(\Omega)\mathrm{s}(\omega) = \mathrm{c}(\Omega)\mathrm{c}(\omega)\}$ and the normalized gauge acceleration magnitude satisfies $\|\mathbf{Q}\|a^4/(\mu r_{\mathrm{eq}}^2) \sim \mathcal{O}(eJ_2)$. This result, however, should be judiciously handled. It is tempting to think that since **Q** consists of constant and periodic terms, the averaged gauge function itself will grow linearly with time. This is generally not true, because

$$\langle \mathbf{q} \rangle \neq \int_t \dot{\bar{\mathbf{q}}} \, \mathrm{d}t. \tag{45}$$

Consequently, in order to satisfy assumption (24), we have to show that the gauge velocity, normalized by some characteristic velocity, is at most of order J_2; otherwise, the above analysis may not hold because the gauge function will violate the smallness assumption. This order-of-magnitude analysis, performed in Section 4.3, shows that the normalized gauge velocity for the current example *is* of order J_2, and hence our enabling assumption holds.

The remaining planetary equations, using the gauge function derivative (44) and the mean non-osculating orbital elements as state variables, are as follows:

$$\dot{\bar{\Omega}} = -\frac{3}{2} J_2 n \left(\frac{r_{\mathrm{eq}}}{p} \right)^2 \cos i, \tag{46}$$

$$\dot{\bar{M}} = \frac{3}{2} J_2 \left(\frac{r_{\mathrm{eq}}}{p} \right)^2 \frac{n}{\sqrt{1-e^2}} [\cos^2 i (e^2 + 4) - 1], \tag{47}$$

where $p = a(1 - e^2)$.

Interestingly, Equation (46) is *identical* to the mean nodal drift equation obtained with the Lagrange gauge ($\mathbf{q} = 0$). The equation for the mean drift of the mean anomaly at epoch calculated using the Lagrange gauge is generally *not* equal to expression (47) and is given by

$$\dot{\bar{M}}_0 = \frac{3}{4} J_2 \left(\frac{r_{\mathrm{eq}}}{p} \right)^2 n \sqrt{1-e^2} (3 \cos^2 i - 1). \tag{48}$$

The gauge symmetry has therefore served as a coordinate transformation which nullified the apsidal line precession at the expense of modifying the perigee passage time. This makes physical sense, as both $\bar{\omega}$ and $\bar{M}_0$ are angles defined in the orbital plane. The spacecraft inertial position is of course invariant to whether the entire orbital plane is rotating or only the timing of perigee passage is changing. Thus, the gauge freedom can be viewed as a transformation from an orbit-fixed frame rotating at rate $\dot{\bar{\omega}}$) to a satellite-fixed frame with the appropriate correction of the mean anomaly rate.

To summarize, we have managed to find a gauge function rate, given by Equation (44), that *identically nullifies* the rate of change of the argument of periapsis in addition to nullifying the mean rates of the eccentricity and inclination. While the line of apsides always regresses or advances (excluding the critical inclination) using osculating orbital elements, the formulation using non-osculating elements yields a dynamical model in which the line of apsides is *stationary*.

In addition, the above representation permits nullification of $\bar{M}_0$ at critical inclination angles that are eccentricity-dependent. To see this, we solve Equation (47) for i and get

$$i_{\text{crit}} = \cos^{-1}\left(\pm\sqrt{\frac{1}{4+e^2}}\right), \tag{49}$$

meaning that for each eccentricity there is a direct and a retrograde orbit for which the only secular change is the drift of the line of nodes.

Furthermore, using the gauge-generalized representation, a polar orbit $(i = 90°)$ constitutes an *invariant manifold* in the non-osculating orbital elements state-space, implying that a polar orbit will show merely a secular change in the periapsis passage time. This invariant manifold does not exist in the mean osculating orbital elements parametrization because in that case a polar orbit will still induce an apsidal line precession.

4.2. NULLIFYING THE NODAL, APSIDAL AND EPOCH RATES

As an additional example, we may eliminate the nodal precession, the apsidal rotation and the epoch variations by nullifying the mean angular rates $\dot{\bar{\Omega}}$, $\dot{\bar{\omega}}$ and $\dot{\bar{M}}_0$, respectively. The resulting equations are Equation (43) and

$$\dot{\bar{\Omega}} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\left\{-3\sin 2i - \left[\frac{\partial\bar{\mathbf{f}}}{\partial i}\right]^{\mathrm{T}}\mathbf{Q}\right\} = 0, \tag{50}$$

$$\dot{\bar{M}}_0 = -\frac{1-e^2}{na^2 e}\left\{\frac{3e(2-3\sin^2 i)}{(1-e^2)} - \left[\frac{\partial\bar{\mathbf{f}}}{\partial e}\right]^{\mathrm{T}}\mathbf{Q}\right\} - \frac{2}{na}\left\{-\frac{3(2-3\sin^2 i)}{a} - \left[\frac{\partial\bar{\mathbf{f}}}{\partial a}\right]^{\mathrm{T}}\mathbf{Q}\right\} \tag{51}$$

Denoting again $\mathbf{Q} = [Q_1, Q_2, Q_3]^{\mathrm{T}}$, the gauge function rate solving the system of equations (43), (50) and (51) is given by

$$\begin{aligned} Q_1 = &-2\mu J_2 r_{\text{eq}}^2\Big(-5e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)\mathrm{c}(\Omega)\mathrm{c}(i)^5 - 2e^2\mathrm{c}(i)^4\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2 \\ &- 2e^2\mathrm{c}(i)^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2 - 2e^2\mathrm{c}(i)^2\mathrm{s}(\omega)\mathrm{c}(\omega)^2\mathrm{c}(\Omega)^2 \\ &+ 6e^2\mathrm{c}(i)^4\mathrm{s}(\omega)\mathrm{c}(\omega)^2\mathrm{c}(\Omega)^2 - e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)^3\mathrm{c}(\Omega)\mathrm{c}(i) \\ &+ 2e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)^3\mathrm{c}(\Omega)\mathrm{c}(i)^3 \\ &+ 2\mathrm{c}(i)^2\mathrm{s}(\omega)\mathrm{c}(\Omega)^2 + 2\mathrm{s}(i)\mathrm{c}(\omega)^3\mathrm{c}(i) + 8\mathrm{s}(i)\mathrm{c}(\omega)\mathrm{c}(i)^3 \\ &- 8\mathrm{s}(i)\mathrm{c}(\omega)^3\mathrm{c}(i)^3 + 2\mathrm{c}(i)^4\mathrm{s}(\omega) - 2\mathrm{c}(i)^2\mathrm{s}(\omega) \\ &- 2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega) + 3e^2\mathrm{c}(i)^2\mathrm{s}(\omega)\mathrm{c}(\Omega)^2 \\ &- 5e^2\mathrm{c}(i)^4\mathrm{s}(\omega)\mathrm{c}(\Omega)^2 + 2\mathrm{c}(\omega)e^2\mathrm{s}(i)\mathrm{c}(i)^3 \\ &- 2\mathrm{c}(\omega)^3e^2\mathrm{s}(i)\mathrm{c}(i)^3 + e^2\mathrm{c}(i)^2\mathrm{s}(\omega)\mathrm{c}(\omega)^2 - 3e^2\mathrm{c}(i)^4\mathrm{s}(\omega)\mathrm{c}(\omega)^2 \end{aligned}$$

$$
\begin{aligned}
&+4\mathrm{s}(i)\mathrm{c}(\omega)\mathrm{c}(i)\mathrm{c}(\Omega)^2\\
&-4\mathrm{s}(i)\mathrm{c}(\omega)^3\mathrm{c}(i)\mathrm{c}(\Omega)^2-16\mathrm{s}(i)\mathrm{c}(\omega)\mathrm{c}(i)^3\mathrm{c}(\Omega)^2\\
&+16\mathrm{s}(i)\mathrm{c}(\omega)^3\mathrm{c}(i)^3\mathrm{c}(\Omega)^2+2\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{s}(\Omega)\mathrm{c}(i)^3-2\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{s}(\Omega)\mathrm{c}(i)^5\\
&-2\mathrm{c}(i)\mathrm{s}(i)\mathrm{c}(\omega)-3e^2\mathrm{c}(i)^2\mathrm{s}(\omega)+5e^2\mathrm{c}(i)^4\mathrm{s}(\omega)-2\mathrm{c}(i)^4\mathrm{s}(\omega)\mathrm{c}(\Omega)^2\\
&+3e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)^3\mathrm{c}(\Omega)\mathrm{c}(i)^5+e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)\mathrm{c}(\Omega)\mathrm{c}(i)\\
&-4\mathrm{c}(\omega)e^2\mathrm{s}(i)\mathrm{c}(i)^3\mathrm{c}(\Omega)^2+4\mathrm{c}(\omega)^3e^2\mathrm{s}(i)\mathrm{c}(i)^3\mathrm{c}(\Omega)^2\\
&+2e^2\mathrm{c}(i)^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)+2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\\
&+8\mathrm{c}(i)^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)-6\mathrm{c}(i)^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\\
&-8\mathrm{c}(i)^4\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\Big)\Big/\Big(4\mathrm{c}(i)\sqrt{1-e^2}a^4e\mathrm{s}(\omega)\\
&\quad(e^4\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)-e^4\mathrm{s}(\Omega)\mathrm{s}(\omega)-\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)\mathrm{s}(\Omega)\mathrm{s}(\omega)\Big)
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
Q_2=&-2\mu J_2r_{\mathrm{eq}}^2\Big(2\mathrm{c}(\Omega)\mathrm{c}(i)^4-2\mathrm{c}(\Omega)\mathrm{c}(i)^2\\
&-3\mathrm{c}(i)^4e^2\mathrm{c}(\Omega)\mathrm{c}(\omega)^2+2\mathrm{c}(i)^2e^2\mathrm{s}(i)\mathrm{s}(\Omega)\mathrm{c}(\omega)^2\\
&+8\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)^3+2\mathrm{s}(\Omega)\mathrm{s}(i)-2\mathrm{s}(i)\mathrm{s}(\Omega)\mathrm{c}(\omega)^2\\
&+3e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)\mathrm{s}(\omega)\mathrm{c}(i)^3-2\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)\\
&+8\mathrm{c}(i)^2\mathrm{s}(i)\mathrm{s}(\Omega)\mathrm{c}(\omega)^2-2\mathrm{c}(i)^2e^2\mathrm{s}(i)\mathrm{s}(\Omega)+\mathrm{c}(i)^2e^2\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\\
&+2\mathrm{s}(\omega)e^2\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)^3\\
&-e^2\mathrm{s}(\Omega)\mathrm{c}(\omega)\mathrm{s}(\omega)\mathrm{c}(i)-8\mathrm{c}(i)^2\mathrm{s}(i)\mathrm{s}(\Omega)-3\mathrm{c}(i)^2e^2\mathrm{c}(\Omega)\\
&+5\mathrm{c}(i)^4e^2\mathrm{c}(\Omega)\Big)\Big/\Big(4\mathrm{c}(i)e(e^4-1)a^4\sqrt{l-e^2}\mathrm{s}(\omega)\Big),
\end{aligned} \tag{53}
$$

$$
\begin{aligned}
Q_3=&\,2\mu J_2r_{\mathrm{eq}}^2(e^2\mathrm{c}(\omega)\mathrm{c}(\Omega)\mathrm{c}(i)^2+e^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(\omega)^2\mathrm{c}(i)\\
&-2\mathrm{s}(\Omega)\mathrm{c}(i)^2\mathrm{c}(\omega)^3e^2\mathrm{s}(i)+5e^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)^3-e^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)\\
&-2e^2\mathrm{c}(i)^3\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\\
&-5e^2\mathrm{c}(\omega)\mathrm{c}(\Omega)\mathrm{c}(i)^4-e^2\mathrm{c}(\omega)^3\mathrm{c}(\Omega)\mathrm{c}(i)^2-3e^2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(\omega)^2\mathrm{c}(i)^3\\
&+2\mathrm{s}(\Omega)\mathrm{c}(i)^2\mathrm{c}(\omega)e^2\mathrm{s}(i)\\
&+3e^2\mathrm{c}(\omega)^3\mathrm{c}(\Omega)\mathrm{c}(i)^4-2\mathrm{s}(\Omega)\mathrm{s}(i)\mathrm{c}(\omega)-2\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i)^4\\
&+2\mathrm{s}(\Omega)\mathrm{s}(\omega)\mathrm{c}(i)^3+2\mathrm{s}(\Omega)\mathrm{s}(i)\mathrm{s}(\omega)^3+2\mathrm{c}(i)\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\\
&-8\mathrm{s}(\Omega)\mathrm{c}(i)^2\mathrm{s}(i)\mathrm{c}(\omega)^3\\
&+8\mathrm{s}(\Omega)\mathrm{c}(i)^2\mathrm{s}(i)\mathrm{c}(\omega)-8\mathrm{c}(i)^3\mathrm{s}(\omega)\mathrm{s}(i)\mathrm{c}(\Omega)\mathrm{c}(\omega)^2\mathrm{s}(i)\\
&\Big/\Big(4(-\mathrm{s}(\Omega)\mathrm{s}(\omega)+\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i))ea^4\sqrt{1-e^2}\mathrm{s}(\omega)\mathrm{c}(i)(e^4-1)\Big).
\end{aligned} \tag{54}
$$

This gauge function rate is regular $\forall\bar{\boldsymbol{\alpha}} \in \mathcal{M}\backslash\{i,\Omega,\omega : \mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i) = s(\Omega)s(\omega)$ or $\omega = k\pi, k = 0,1,2\}$ and satisfies $\|\mathbf{Q}\|a^4/(\mu r_{\mathrm{eq}}^2) \sim \mathcal{O}(J_2)$. The remaining planetary equations are

$$\dot{\bar{i}} = \frac{3}{4}J_2\left(\frac{r_{\mathrm{eq}}}{p}\right)^2 n\tan\omega\sin 2i, \tag{55}$$

$$\dot{\bar{e}} = -\frac{3}{2}J_2\left(\frac{r_{\mathrm{eq}}}{p}\right)^2 n\frac{f_1}{f_2}\frac{\tan i}{\sin\omega}, \tag{56}$$

where

$$\begin{aligned} f_1 =& [\mathrm{c}^3(i)\mathrm{s}(\omega)\mathrm{c}(\Omega)\mathrm{c}(\omega) - \mathrm{s}(\Omega)\mathrm{c}^2(i) + \mathrm{c}^2(i)\mathrm{s}(\Omega)\mathrm{c}^2(\omega)]e^2 - 4\mathrm{s}(\Omega)\mathrm{c}(^2i) \\ &+ 4\mathrm{c}^3(i)\mathrm{s}(\omega)\mathrm{c}(\Omega)\mathrm{c}(\omega) + 4\mathrm{c}^2(i)\mathrm{s}(\Omega)\mathrm{c}^2(\omega) + \mathrm{s}(\Omega) \\ &- \mathrm{s}(\omega)\mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i) - \mathrm{s}(\Omega)\mathrm{c}^2(\omega), \end{aligned} \tag{57}$$

$$f_2 = -\mathrm{s}(\Omega)\mathrm{s}(\omega) + \mathrm{c}(\Omega)\mathrm{c}(\omega)\mathrm{c}(i). \tag{58}$$

Equation (55) is singular at $\omega = \pi/2$ and $\omega = 3\pi/2$. Equation (56), on the other hand, is regular at $\omega = k\pi$, $k = 0,1,2$ and $c(\Omega)c(\omega)c(i) = s(\Omega)s(\omega)$, since $\lim_{\omega\to k\pi} f_1/\sin\omega = e^2\cos^3 i\cos\Omega - \cos\Omega\cos i + 4\cos^3 i\cos\Omega$; however, the gauge function rate at these points is infinite. Therefore, there are no admissible values of orbital elements that can nullify either of Equations (55), (56) (the equatorial case is excluded due to singularity of the Poisson matrix), implying that non-osculating eccentricity and inclination variations are always required to maintain a stationary node, perigee and epoch.

Similarly to the discussion in Section 4.1, the nullification of the nodal, apsidal and epoch rates has been carried out assuming that the normalized gauge velocity is, at most, of order J_2. This assumption is verified in the following section.

4.3. ORDER OF MAGNITUDE ANALYSIS FOR THE GAUGE VELOCITY

In this section, we shall present an analysis of the order of magnitude of the gauge velocity, and show that the normalized gauge velocity is *at most* of order J_2. Since order of magnitude of the normalized gauge velocity does not exceed the order of magnitude of the normalized perturbing potential, the separation of time scales leading to the first-order averaging holds.

To start, recall that the mean gauge velocity $\bar{\mathbf{q}}$, defined in Equation (14), is given by

$$\bar{\mathbf{q}} = J_{\mathbf{f}}(\bar{\boldsymbol{\alpha}})\dot{\bar{\boldsymbol{\alpha}}}, \tag{59}$$

where $J_{\mathbf{f}} = \partial \mathbf{f}/\partial \bar{\boldsymbol{\alpha}}$ and $\dot{\bar{\boldsymbol{\alpha}}} = [\dot{\bar{a}}, \dot{\bar{e}}, \dot{\bar{i}}, \dot{\bar{\Omega}}, \dot{\bar{\omega}}, \dot{\bar{M}}_0]^{\mathrm{T}}$. The expression for $J_{\mathbf{f}}$ is given by Equation (31). In the remainder of this section, we shall again drop the bar and treat the orbital elements as averaged quantities.

The example given in Section 4.1 nullified the mean rotation rate of the apsi-dal line (argument of perigee rate) in addition to nullifying the mean rates of the semi-major axis, eccentricity and inclination. The resulting nodal precession and drift of the mean anomaly were given by Equations (46) and (47). Substituting these expressions into (59) gives

$$\bar{\mathbf{q}} = J_2 \frac{9}{4}\left(\frac{r_{\mathrm{eq}}}{p}\right)^2 nae \cos i \begin{bmatrix} -\sin\Omega\cos\omega - \cos\Omega\sin\omega\cos i \\ \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i \\ 0 \end{bmatrix}. \tag{60}$$

The magnitudes of the trigonometric functions are upper-bounded by 1, and thus cannot alter the order of magnitude of the gauge velocity in (60). Thus, the normalized gauge velocity magnitude satisfies

$$\mathcal{O}\left(\frac{\|\bar{\mathbf{q}}\|}{na}\right) \sim eJ_2 \leqslant J_2 \tag{61}$$

and hence the analysis in the Section 4.1 holds *for all* times, since the original time scale separation assumption (i.e. $\dot{f} \gg \dot{\Omega}, \dot{\omega}, \dot{M}_0$), enabling the averaging analysis due to the J_2 perturbation is not violated. To illustrate this issue, consider the following example:

EXAMPLE. Consider an eccentric Earth orbit with low perigee of 800 km, $i = 20°$, $\Omega = 0$, $\omega = 90°$ and $e = 0.1$. Substitution into Equation (60) gives

$$\frac{\|\bar{\mathbf{q}}\|}{na} = 1.4027 \times 10^{-4}. \tag{62}$$

For comparison, let us examine the order of magnitude of the J_2 averaged normalized perturbing potential:

$$\frac{R}{n^2 r_{\mathrm{eq}}^2} = \frac{J_2}{4(1-e^2)^{3/2}}(2 - 3\sin^2 i) = 4.5285 \times 10^{-4}. \tag{63}$$

Consequently, not only that the gauge velocity is at most of order J_2, but moreover, it is *smaller* than the normalized perturbing potential.

A similar analysis of the example in Section 4.2, nullifying the nodal, apsidal and epoch rates in addition to the semi-major axis rate, yields:

$$\bar{\mathbf{q}} = J_2 \frac{9}{4}\left(\frac{r_{\mathrm{eq}}}{p}\right)^2 an \begin{bmatrix} \frac{f_1 \tan i(-\cos i \sin\Omega \sin\omega + \cos\omega\cos\Omega)}{f_2 \sin\omega} & -e\frac{\sin\Omega\sin\omega\sin i\tan\omega\sin 2i}{2} \\ \frac{f_1 \tan i(\cos i\cos\Omega\sin\omega+\cos\omega\sin\Omega)}{f_2\sin\omega} & +e\frac{\cos\Omega\sin\omega\sin i\tan\omega\sin 2i}{2} \\ \frac{f_1\tan i\sin i}{f_2} - \frac{e\sin\omega\cos i\tan\omega\sin 2i}{2} & \end{bmatrix} \tag{64}$$

where $f_1(\Omega, \omega, i)$ and $f_2(\Omega, \omega, i)$ are bounded functions given by Equations (57) and (58), respectively. Thus, similarly to Equation (61), we have

$$\mathcal{O}\left(\frac{\|\bar{\mathbf{q}}\|}{na}\right) \sim J_2, \tag{65}$$

so the gauge velocity components are of order J_2, and the underlying assumption holds for this example as well.

5. Conclusions

This paper developed averaged planetary equations using non-osculating orbital elements. The excess freedom obtained by transforming from the inertial space to the orbital elements space, termed gauge freedom, was utilized to nullify four planetary equations.

The gauge freedom considerably broadens our understanding of the non-Keplerian dynamics. While thus far it has been widely assumed that osculating orbital elements constitute the most advantageous representation of perturbed Keplerian dynamics, this paper shows that there are cases where it is more beneficial to use non-osculating elements.

The gauge formalism may be interpreted as a coordinate transformation, since it represents an inherent symmetry that exists in the Lagrange planetary equations. The investigation of this symmetry is a subject of future research, as well as investigation of short-period effects using gauge-generalized planetary equations and the incorporation of high-order zonal harmonics.

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References

Bate, R. R., M. D. D. and White, J. E.: 1971, *Fundamentals of Astrodynamics*, Dover.

Battin, R. H.: 1999, *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA, Reston, VA.

Brouwer, D. and Clemence, G. M.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York.

Efroimsky, M.: 2002, 'Equations for the orbital elements. hidden symmetry', *Preprint No 1844 of the Institute of Mathematics and its Applications, University of Minnesota* (http://www.ima.umn.edu/preprints/feb02/1844.pdf; -astroph/0212245).

Efroimsky, M. and Goldreich, P.: 2003, 'Gauge symmetry of the N-body problem in the Hamilton-Jacobi approach', *J. Math. Phys.* **44**, 5958–5977.

Efroimsky, M. and Goldreich, P.: in press, 'Gauge freedom in the N-body problem of celestial mechanics', *Astron. & Astrophys.*

Euler, L.: 1748, *Opera Mechanica et Astronomica*, Birkhauser-Verlag, Switzerland.

Hadjifotinou, K. G.: 2000, 'Numerical integration of satellite orbits around an oblate planet', *Astron. Astrophys.* **354**, 328–333.

Koon, W. S., M. J. E. and Murray R. M.: 2001, 'J_2 Dynamics and Formation Flight'. In: *Proceedings of the 2001 AIAA Guidance, Navigation, and Control Conference*. Montreal, Quebec.

Lagrange, J. L.: 1809, 'Sur la Théorie Générale de la Variation des Constants Arbitraries dans tous les Problèmes de la Méchanique', *Lu le 13 mars 1809 à 1'Institut de France*.

Newman, W. and Efroimsky, M. 2003, 'The method of variation of constants and multiple time scales in orbital mechanics', *Chaos* **13**, 476–485.

Palmer, P. L. and Hashida, Y.: 2001, 'Epicyclic motion of satellites about an oblate planet', *J. Guidance, Control Dynam.* **24**(3), 586–596.

Schaub, H. and Alfriend, K. T.: 2001, 'J_2 invariant relative orbits for spacecraft formations', *Celest. Mech. Dynam. Astron.* **79**, 77–95.

Seager, S. and Hui, L. 2002, 'Constraining the rotation rate transiting extra-solar planets by oblateness measurements', *Astrophys. J.* **574**, 1004–1010.