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Gauge theory for finite-dimensional dynamical systems

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Gauge theory is a well-established concept in quantum physics, electrodynamics, and cosmology. This concept has recently proliferated into new areas, such as mechanics and astrodynamics. In this paper, we discuss a few applications of gauge theory in finite-dimensional dynamical systems. We focus on the concept of rescriptive gauge symmetry, which is, in essence, rescaling of an independent variable. We show that a simple gauge transformation of multiple harmonic oscillators driven by chaotic processes can render an apparently "disordered" flow into a regular dynamical process, and that there exists a strong connection between gauge transformations and reduction theory of ordinary differential equations. Throughout the discussion, we demonstrate the main ideas by considering examples from diverse fields, including quantum mechanics, chemistry, rigid-body dynamics, and information theory. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2720098]

Gauge theories in physics constitute a fundamental tool for modeling the electromagnetic, weak, and strong forces. They have been used in a variety of fields, ranging from sub-atomic physics to cosmology. The basic mathematical tool generating the gauge theories is symmetry; i.e., a redundancy in the description of the system. In physics, such redundancy emerges whenever the number of the mathematical variables exceeds the number of the physical degrees of freedom (DOF). This excess immediately entails internal freedom; i.e., invariance of the physical sector of the theory under a group of transformations. Although symmetries have long been recognized as a fundamental tool for solving ordinary differential equations, they have not been always formally categorized as gauge theories. In this paper, we show how simple systems described by ordinary differential equations exhibit gauge symmetry, and discuss a few practical applications of this approach. In particular, we utilize the notion of gauge symmetry to question some common engineering misconceptions of chaotic and stochastic phenomena, and show that seemingly "disordered" (deterministic) or "random" (stochastic) behaviors can be "ordered." This brings into play the notion of *observation*; we show that temporal observations may be misleading when used for chaos detection.

I. INTRODUCTION

In modern physics, gauge theories are the most powerful methods for understanding interactions among fields. The two major successes of the gauge approach in particle physics were the creation of the unified electro-weak model and of the strong-force theory known as chromodynamics. The further unification of the fundamental forces, attempted within the string theories, also preserves the gauge structure. The linearized version of general relativity (GR) also contains some form of gauge freedom. While Einstein's GR itself is not a gauge theory, one of the alternative versions of the gravitation theory is. The Yang-Mills construction, the standard approach to field theory, is a particular example of gauge theories with non-Abelian symmetry groups.

The question to be raised at this point is: Can the gauge-theoretical approach be applied to finite-dimensional systems, and with what degree of success? Keeping in mind that Lie symmetry has been a major tool in the study of ordinary differential equations (ODEs),[1] the answer must be positive; however, we have not established yet a clear-cut connection between Lie symmetry and gauge symmetry. We plan to do so in the sequel, and show that, indeed, Lie point-symmetry is closely connected to gauge symmetry, albeit this connection is not always straightforward. In order to facilitate the establishment of such a connection, we use gauge symmetry in a more general context; i.e., a context of a symmetry defined by diffeomorphisms. This will ultimately allow us to combine various manifestations of gauge symmetry in finite-dimensional dynamical systems under a single mathematical realm.

Application of gauge transformations to partial differential equations (PDEs) is not an exclusive monopoly of the field-theory scholars. This technique was introduced into the elasticity theory by Kunin.[2] In the ODE context, it was pioneered by Efroimsky[3] and then employed in Refs. 4–8 to develop a gauge-generalized astrodynamical theory for modeling the effect of orbital perturbations using non-osculating orbital elements. Efroimsky's gauge theory, however, does not deal with scale transformations, while Kunin's gauge theory is concerned mostly with local gauge groups and discrete symmetries. Thus far, there has not been a unified gauge theory that is able to support both Yang-Mills-like gauge theories and scaling theories in finite-dimensional systems.

In the current paper, we shall attempt to fill this gap by developing a gauge theory for finite-dimensional dynamical systems through two gauge symmetry mechanisms; the first symmetry mechanism will be called *rescriptive gauge sym-*

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metry (the definition of *rescriptive* is given in Webster's Revised Unabridged Dictionary (1913): "Pertaining to, or answering the purpose of, a rescript; hence, deciding; settling; determining), evoked by carrying out a *rescriptive gauge transformation*.

Rescriptive gauge symmetry succumbs to the fundamental notion of gauge transformations, namely, a change of scale, and is also intimately connected to *bilinearity*. To show rescriptive gauge symmetry, we shall carry out an infinitesimal transformation of the independent variable—which in the bulk of our subsequent discussion will be time—into a different scale. The manifestation of gauge symmetry in this case will be reflected in the ability to obtain equivalence between the direction fields of the original and gauge-transformed systems. In many practical applications, this implies that the system can be reduced to a form amenable for quadrature (e.g., linear ODEs). We shall formalize this observation by establishing a mechanism for *reduction* through rescriptive gauge symmetry.

To illustrate the concept of rescriptive gauge symmetry, we will present a number of physical examples taken from diverse scientific and engineering fields, including rigid-body dynamics, finite-dimensional quantum mechanical systems, chemistry, and information theory.

An instrumental constituent of our new theory is the *gauged pendulum*. Generally speaking, a gauged pendulum is a physical system with a quadratic integral of motion, whose behavior in the time domain can be arbitrary, although its phase-space structure remains invariant under a change of scale. This implies that after a suitable scale transformation, harmonic oscillations will emerge. We show that many physical systems can be either reformulated to match the formalism of the gauged pendulum, or are natural gauge pendulums *per se*; a classical example for a natural gauged pendulum is the Euler-Poinsot system, to be analyzed subsequently.

We ultimately utilize the notion of a gauged pendulum to question some common engineering misconceptions of chaotic and stochastic phenomena, and show that seemingly "disordered" (deterministic) or "random" (stochastic) behaviors can be "ordered," or, put differently, can evoke simple patterns[9,10] using an infinitesimal transformation of the time scale. This brings into play the notion of *observation* and *observables*; we show that temporal observations may be misleading when used for chaos detection.

II. PRELIMINARIES AND DEFINITIONS

Consider the following finite-dimensional dynamical systems:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{1}$$

$$\frac{d\mathbf{x}}{dt} = \mathbf{g}(\mathbf{x}), \tag{2}$$

where $\mathbf{x} \in \mathbb{R}^n$, and $\mathbf{f}, \mathbf{g}: \mathbb{R}^n \to \mathbb{R}^n$.

If $\mathbf{g}(\mathbf{x})$ happens to be equal to $\mathbf{f}(\mathbf{x})$ multiplied by some nonvanishing G, then (2) may be trivially rewritten as

$$\frac{d\mathbf{x}}{d\tau} = \mathbf{f}(\mathbf{x}), \quad d\tau = G dt. \tag{3}$$

The dynamics $\mathbf{x}(t)$ furnished by (2) differs from the dynamics $\mathbf{x}(t)$ rendered by (1). At the same time, the functional form of the solution $\mathbf{x}(t)$ to (1) coincides with the functional form of the solution $\mathbf{x}(\tau)$ to (3). Loosely speaking, the solution to (2) mimics that of (1), though with respect to a rescaled time τ. This fact can be cast also in the following form: the qualitative features of the motion governed by (2) will coincide with those of the motion given by (1): these systems will have coinciding integrals of motion and identical topological structures of the phase portraits (as long as G remains nonvanishing).

Now, let us ask what this G may be. It may be an arbitrary nonvanishing function of $\mathbf{x}$ or/and t. More generally, it may also depend upon some extra variables $\mathbf{u}$, which in their turn may be functions of $\mathbf{x}$ or/and t. Moreover, $\mathbf{u}$ may also depend upon some completely extrinsic variables $\mathbf{y}$. Finally, these extrinsic variables may be stochastic or, for example, chaotic. In the latter case, (2) will be a part of a much more sophisticated system that may include chaos. Still, the $\mathbf{x}(\tau)$ sector of that system will remain qualitatively equivalent to the simpler dynamics (1) and will preserve the same integrals. In particular, if (1) has no chaos in it, then the $\mathbf{x}(\tau)$ sector of the sophisticated system involving (2) will not have it either, even though it may seem to be chaotic when $\mathbf{y}$ brings chaos into the time-rescaling formula.

Let us formalize the above observation using the concept of *rescription*. A rescription operator in the time domain, i.e., $\mathfrak{F}_t$, acts upon the vector field $\mathbf{f}$ in the following manner:

$$\mathfrak{F}_t \circ \mathbf{f} = \mathbf{g}(\mathbf{x}, \mathbf{u}(\mathbf{x}, t)) = \frac{d\mathbf{x}}{dt}. \tag{4}$$

The vector field $\mathbf{u}: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, where $m \leq n$, will be called a *rescriptor*. The rescriptor may be either *static*, i.e., will depend upon $\mathbf{x}$ and/or t only, or *dynamic*. In the latter case, the rescriptor itself constitutes a dynamical system of the general form

$$\frac{d\mathbf{u}}{dt} = \mathbf{h}(\mathbf{x}, d\mathbf{x}/dt, \mathbf{y}, \mathbf{u}), \tag{5a}$$

$$\frac{d\mathbf{y}}{dt} = \mathbf{h}_1(\mathbf{y}, \mathbf{u}), \tag{5b}$$

where $\mathbf{y} \in \mathbb{R}^q$ and $\mathbf{h}_1: \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^q$. By restricting the dynamic rescriptor from dependence on $\mathbf{u}$, we can unify the dynamic and static rescription; viz., a static rescription can be defined as a special case of a dynamic rescription.

In most cases, the rescription modifies—sometimes intentionally, such as in the case of control inputs—the fundamental properties of the original system. These "fundamental properties" may be, for instance, integrability, symmetry, and structure/volume-preserving measures. The new properties of the rescribed system can be investigated both in the time domain and in the phase space.

However, in some cases the rescription is merely an illusion, that is, the rescription does not change the phase space and the fundamental properties of the original system, although it could modify the flow $\varphi(x_i(t=t_0),t)$. In this general setting, the system is invariant under the action of some (possibly time-varying) finite-dimensional *gauge group* $\mathcal{G}$.

III. RESCRIPTIVE GAUGE SYMMETRY

We ask whether the system can be "de-rescribed" by finding new independent variables, i.e., τ_j, possibly different for each rescriptor component u_i, satisfying

$$d\tau_j = G_i(\mathbf{x},u_i(\mathbf{x},t),d\mathbf{x},dt), \quad (6)$$

for which

$$\mathfrak{F}_\tau \circ \mathbf{g} = \mathbf{f}(\mathbf{x}) = \mathbf{x}', \quad (7)$$

where the operator $()'$ denotes differentiation of each x_i with respect to some τ_j,

$$\mathbf{x}' = \frac{dx_i}{d\tau_j}, \quad i=1,\ldots,n, \quad j\in[1,\ldots,n]. \quad (8)$$

If $\exists d\tau_j$, $j\in[1,\ldots,n]$ satisfying (6) such that (7) holds, then we shall say that system (4) exhibits *full rescriptive gauge symmetry* under the *rescriptive gauge transformation* (6). In this case, $\mathbf{u}$ becomes either a static or a dynamic *rescriptive gauge function*.

A *rescriptive gauge symmetry of order p* or simply *partial rescriptive gauge symmetry* comes about when the rescriptive gauge transformation de-rescribes only p state variables, $p<n$; viz.,

$$\mathfrak{F}_\tau \circ g_i = f_i(\mathbf{x}) = x_i', \quad i\in\mathbb{N}^p. \quad (9)$$

In this case, if $t\in \mathfrak{t}=[0,t_f)$, $t_f\le\infty$ and $\tau_j\in\mathbb{R}$, $j\in[1,\ldots,p]$, then $\exists \tau_{j0},t_0,x_i(t_0),x_i(\tau_{j0})$, such that the flow satisfies

$$\varphi(x_i(t_0),t) = \varphi(x_i(\tau_{j0}),\tau_j), \quad (10)$$

for $\mathfrak{t}\cap\mathbb{R}$, where the flow is interpreted as the one-parameter group of transformations

$$G_t{:}x_i(t_0)\to x_i(t), \quad G_\tau{:}x_i(\tau_0)\to x_i(\tau). \quad (11)$$

The notion of rescriptive gauge symmetry can be applied for "ordering" seemingly "disordered" phenomena, and for solving ordinary differential equations (ODEs). We shall illustrate these ideas by discussing a few examples of practical interest. We embark on our quest by presenting the notion of a *gauged pendulum*, dwelt upon in the following subsection.

A. The gauged pendulum

Finite-dimensional systems can often be modeled by Hamiltonian vector fields induced by a nominal Hamiltonian $\mathcal{H}$ and a perturbing Hamiltonian $\Delta\mathcal{H}$. Moreover, in ubiquitous fields of science and engineering, $\mathcal{H}$ is comprised of n uncoupled harmonic oscillators,[11] namely,

$$\mathcal{H}[\mathbf{q}(t),\mathbf{p}(t)] = \frac{1}{2}(\mathbf{p}^T\mathbf{p}+\mathbf{q}^T\Omega\mathbf{q}) = \frac{1}{2}\sum_{i=1}^{n}(p_i^2+\omega_i^2q_i^2) = \frac{1}{2}\sum_{i=1}^{n}\mathcal{H}_i[q_i(t),p_i(t)], \quad (12)$$

where $\Omega=\mathrm{diag}(\omega_1^2,\ldots,\omega_n^2)$,

$$\mathbf{q}=[q_1,\ldots,q_n]^T, \quad \mathbf{p}=[p_1,\ldots,p_n]^T, \quad (13)$$

are the generalized coordinates and conjugate momenta, respectively. Hamilton's equations for $i=1,\ldots,n$ are then

$$\dot{q}_i = p_i, \quad (14a)$$

$$\dot{p}_i = -\omega_i^2 q_i. \quad (14b)$$

Carrying out the point transformation into action-angle variables, given by

$$q_i = \sqrt{\frac{\Phi_i}{\omega_i}}\sin\phi_i, \quad p_i = \sqrt{\Phi_j\omega_j}\cos\phi_i, \quad (15)$$

simplifies the Hamiltonian (12) even further, into

$$\mathcal{H}[\boldsymbol{\omega}(t),\boldsymbol{\Phi}(t)] = \frac{1}{2}\boldsymbol{\omega}^T\boldsymbol{\Phi} = \frac{1}{2}\sum_{i=1}^{n}\omega_i\Phi_i, \quad (16)$$

where

$$\boldsymbol{\omega}=[\omega_1,\ldots,\omega_n]^T, \quad \boldsymbol{\Phi}=[\Phi_1,\ldots,\Phi_n]^T. \quad (17)$$

To proceed, let us choose an arbitrary nonvanishing (not necessarily smooth) scalar field $u_i(\mathbf{q},\mathbf{p})$ to serve as our rescriptor, coupling the dynamics of the n pendulums, and rewrite (14) into the *strictly bilinear form* (a strictly bilinear system with respect to $\mathbf{x}$ and $\mathbf{u}$ has the structure $\dot{\mathbf{x}}=M\mathbf{x}\mathbf{u}$, $M\in\mathbb{R}^{n\times n}$; see Ref. 12 for details) in $[q,p]$ and u:

$$\dot{q}_i = p_iu_i(\mathbf{q},\mathbf{p}), \quad (18a)$$

$$\dot{p}_i = -\omega_i^2q_iu_i(\mathbf{q},\mathbf{p}). \quad (18b)$$

Obviously, a constant of motion for each of the pairs (q_i,p_i) would be

$$\mathcal{C}_i = \frac{1}{2}(p_i^2+\omega_i^2q_i^2), \quad i=1,\ldots,n, \quad (19)$$

although $\mathcal{C}_i$ is no longer the Hamiltonian. Nevertheless, system (18) remains integrable regardless of the particular form of u_i, since there are n integrals for n degrees of freedom. This can be readily observed by performing the (affine in dt) rescriptive gauge transformation

$$d\tau_i = u_i(\mathbf{p},\mathbf{q})dt, \quad (20)$$

which, on the one hand, extends (18) into the state-space model

$$\dot{q}_i = p_i u_i(\mathbf{p},\mathbf{q}), \tag{21a}$$

$$\dot{p}_i = -\omega_i^2 q_i u_i(\mathbf{p},\mathbf{q}), \tag{21b}$$

$$\dot{\tau}_i = u_i(\mathbf{p},\mathbf{q}), \tag{21c}$$

but, on the other hand, transforms (18) back into the simple harmonic oscillator form in the independent variables τ_i, assuming the "symplectic" structure

$$q_i' = p_i, \tag{22a}$$

$$p_i' = -\omega_i^2 q_i. \tag{22b}$$

Thus, u_i is a rescriptive gauge function $\forall i$, and $\mathcal{C}_i$ can be interpreted as the Hamiltonian again; that is,

$$\mathcal{H}[\mathbf{q}(\tau_i),\mathbf{p}(\tau_i)] = \frac{1}{2}\sum_{i=1}^{n} C_i = \frac{1}{2}\sum_{i=1}^{n}(p_i^2 + \omega_i^2 q_i^2). \tag{23}$$

In this example, the transformation $t \mapsto \tau_i, i=1,\ldots,n$ is therefore a static rescriptive gauge transformation. This means that u_i may be used to control the flow of p_i and q_i in the time domain, but the persistence of the integrability under the transformation (20) forces the system to exhibit the same behavior as the harmonic oscillator in the modified times τ_i for each degree of freedom.

In order to generalize this concept and to illustrate how rescriptive gauge functions emerge in common physical systems, we must allow u_i to be an output of a dynamical system, giving rise to dynamic rescription, defined in Sec. II. In this case, system (18), written for each degree of freedom, $i=1,\ldots,n$, becomes

$$\dot{q}_i = p_i u_i(\mathbf{p},\mathbf{q}), \tag{24a}$$

$$\dot{p}_i = -\omega_i^2 q_i u_i(\mathbf{p},\mathbf{q}), \tag{24b}$$

$$\dot{u}_i = h_i(\mathbf{p},\mathbf{q},u_i,\mathbf{y}), \tag{24c}$$

$$\dot{\mathbf{y}} = \mathbf{h}_1(\mathbf{u},\mathbf{y}). \tag{24d}$$

Carrying out the rescriptive gauge transformation (20) reveals a partial rescriptive gauge symmetry:

$$q_i' = p_i, \tag{25a}$$

$$p_i' = -\omega_i^2 q_i, \tag{25b}$$

$$u_i' = \frac{1}{u_i} h_i(\mathbf{p},\mathbf{q},u_i,\mathbf{y}). \tag{25c}$$

Thus, independently of the particular characteristics of the dynamic (or static) rescriptive gauge function, i.e., u_i, the system re-assumes the harmonic oscillator structure for (q_i,p_i). This situation can therefore be viewed as a generalization of the pendulum model. The persistence of the harmonic oscillations under the rescriptive gauge transformation gives rise to the concept of a *gauged pendulum*. The gauged pendulum is a dynamical system whose flow becomes periodic under the rescriptive gauge transformation, although the flow of the original system may exhibit arbitrary behavior in the time domain. Such systems arise in ubiquitous fields of science and engineering. For example, the following model arises in the study of quantum mechanical phenomenon (assuming a zero decoherence coefficient):[13]

$$\dot{r}_1 = -u_1(r_1,r_2)u_2(r_1,r_2)r_2, \tag{26}$$

$$\dot{r}_2 = u_1(r_1,r_2)u_2(r_1,r_2)r_1. \tag{27}$$

This is obviously a gauged pendulum with the static rescriptor $u=u_1u_2$. We shall subsequently dwell upon additional physical examples.

An alternative formulation of systems exhibiting partial rescriptive gauge symmetry with a dynamic rescriptive gauge function may written as

$$\dot{q}_i = p_i u_i(\mathbf{p},\mathbf{q}), \tag{28a}$$

$$\dot{p}_i = -\omega_i^2 q_i u_i(\mathbf{p},\mathbf{q}), \tag{28b}$$

$$\dot{u}_i = h_i(\mathbf{p},\mathbf{q},u_i), \tag{28c}$$

which becomes

$$q_i' = p_i, \tag{29a}$$

$$p_i' = -\omega_i^2 q_i, \tag{29b}$$

$$u_i' = \frac{1}{u_i} h_i(\mathbf{p},\mathbf{q},u_i), \tag{29c}$$

after de-rescription using our standard rescriptive gauge transformation. Here, the rescriptor u_i still constitutes a dynamic rescriptive gauge, albeit it is no longer an output of an auxiliary dynamical system. In fact, if we relieve h_i from direct dependence upon u_i, viz., $u_i=h_i(\mathbf{p},\mathbf{q})$, then we uncover additional integrals of the motion, i.e., $\mathcal{K}_i$, defined by the quadrature

$$\mathcal{K}_i = \frac{1}{2}u_i^2 - \int h_i[\mathbf{q}(\tau_i),\mathbf{p}(\tau_i)]d\tau_i. \tag{30}$$

These new constants possess a clear meaning, revealed by writing

$$\dot{u}_i = -\frac{\partial \mathcal{K}_i}{\partial \tau_i} = h_i, \tag{31a}$$

$$\dot{\tau}_i = \frac{\partial \mathcal{K}_i}{\partial u_i} = u_i. \tag{31b}$$

Hence, τ_i and u_i can be interpreted as generalized coordinates and conjugate momenta, respectively, evolving on a $2n$-dimensional symplectic manifold. $\mathcal{K}_i$ is then a Hamiltonian, and the dynamics of (u_i,τ_i) is integrable. This observation is not limited to Hamiltonians of the form (23); rather, if $\mathcal{H}_i(p_i,q_i)=\text{const.}$ is a given Hamiltonian, then Hamilton's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}_i(p_i,q_i)}{\partial p}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}_i(p_i,q_i)}{\partial q}, \tag{32}$$

undergoing a rescription

$$\dot{q}_i = \frac{\partial \mathcal{H}_i(p_i,q_i)}{\partial p} u_i(\mathbf{p},\mathbf{q}), \quad \dot{p}_i = -\frac{\partial \mathcal{H}_i(p_i,q_i)}{\partial q} u_i(\mathbf{p},\mathbf{q}), \tag{33}$$

will still possess $\mathcal{H}_i(p_i,q_i)=\text{const.}$ as an integral, and can be de-rescribed using the rescriptive gauge transformation $d\tau = u_i(\mathbf{p},\mathbf{q})dt$ into

$$q_i' = \frac{\partial \mathcal{H}_i(p_i,q_i)}{\partial p}, \quad \dot{p}' = -\frac{\partial \mathcal{H}_i(p_i,q_i)}{\partial q}. \tag{34}$$

Finally, a slightly different formulation of the gauged pendulum with a dynamics rescriptive gauge symmetry, to be illustrated in Sec. III B, may be written as

$$\dot{q}_i = k_1 p_i u_i(\mathbf{p},\mathbf{q}), \tag{35a}$$

$$\dot{p}_i = k_2 q_i u_i(\mathbf{p},\mathbf{q}), \tag{35b}$$

$$\dot{u}_i = h_i(\mathbf{p},\mathbf{q},u_i), \tag{35c}$$

which becomes

$$q_i' = k_{1i} p_i, \tag{36a}$$

$$p_i' = k_{2i} q_i, \tag{36b}$$

$$u_i' = \frac{1}{u_i} h_i(\mathbf{p},\mathbf{q},u_i), \tag{36c}$$

after de-rescription using the rescriptive gauge transformation $\tau_i = u_i dt$. Here, we have

$$\mathcal{C}_i = \frac{1}{2}(k_{1i} p_i^2 - k_{2i} q_i^2), \quad i = 1, \ldots, n, \tag{37}$$

as integrals. However, an important caveat is that (36a) and (36b) may be viewed as a gauged pendulum only if $k_{1i}k_{2i} < 0$. Otherwise, the de-rescription will yield hyperbolic motion in the variable τ.

B. Eulerian systems

In a body-fixed frame, the attitude dynamics of a rigid body are usually formulated by means of the *Euler-Poinsot* equations. In a free-spin case, these equations look like

$$\mathbb{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbb{I}\boldsymbol{\omega} = 0, \tag{38}$$

$\mathbb{I}$ being the inertia tensor and $\boldsymbol{\omega} = [\omega_1, \ \omega_2, \omega_3]^T \in \mathfrak{S}$ the body angular velocity vector, where $\mathfrak{S}$ is the foliation $\{(I\omega_1, I\omega_2, I\omega_3) | I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = G^2\}$, G being the total angular momentum.

Assuming that the body axes coincide with the principal axes of inertia,

$$\mathbb{I} = \text{diag}(I_1, I_2, I_3), \tag{39}$$

the Euler-Poinsot equations are

$$\dot{\omega}_1 = \sigma_1 \omega_2 \omega_3, \tag{40a}$$

$$\dot{\omega}_2 = \sigma_2 \omega_1 \omega_3, \tag{40b}$$

$$\dot{\omega}_3 = \sigma_3 \omega_1 \omega_2, \tag{40c}$$

where

$$\sigma_1 = \frac{I_2 - I_3}{I_1}, \quad \sigma_2 = \frac{I_3 - I_1}{I_2}, \quad \sigma_3 = \frac{I_1 - I_2}{I_3}. \tag{41}$$

We shall now show that the Euler-Poinsot equations are a classical example of the gauged pendulum concept with a dynamic rescriptive gauge, exhibiting partial rescriptive gauge symmetry of order 2. To that end, define the rescriptive gauge transformation

$$d\tau = \omega_3 dt \tag{42}$$

and rewrite (40) into

$$\omega_1' = \sigma_1 \omega_2, \tag{43a}$$

$$\omega_2' = \sigma_2 \omega_1, \tag{43b}$$

$$\omega_3' = \frac{\sigma_3}{\omega_3} \omega_1 \omega_2, \tag{43c}$$

which adheres to the gauged pendulum model (36). Thus, in the modified scale τ, ω_1 and ω_2 will exhibit harmonic oscillations with frequency $\sqrt{|\sigma_1\sigma_2|}$ if $\sigma_1\sigma_2 < 0$, given by

$$\omega_2(\tau) = \frac{-\sigma_2\omega_{10}\sin(\omega_0\tau_0) + \omega_{20}\omega_0\cos(\omega_0\tau_0)}{\omega_0}\cos(\omega_0\tau) + \frac{\sigma_2\omega_{10}\cos(\omega_0\tau_0) + \omega_{20}\omega_0\sin(\omega_0\tau_0)}{\omega_0}\sin(\omega_0\tau), \tag{44}$$

$$\omega_1(\tau) = \frac{\sigma_2\omega_{10}\cos(\omega_0\tau_0) + \omega_{20}\omega_0\sin(\omega_0\tau_0)}{\sigma_2}\cos(\omega_0\tau) - \frac{\omega_{20}\omega_0\cos(\omega_0\tau_0) - \sigma_2\omega_{10}\sin(\omega_0\tau_0)}{\sigma_2}\sin(\omega_0\tau), \tag{45}$$

where $\omega_0 = \sqrt{|\sigma_1\sigma_2|}$, $\omega_{10} = \omega_1(\tau_0)$, $\omega_2(\tau_0) = \omega_{20}$.

The solution for the dynamic rescriptor ω_3 can now be easily solved by quadrature. (Note that here the rescriptor has units of angular velocity, while in the Newtonian case it was the velocity. We shall reiterate on this issue in the following sections.) Since

$$C = \frac{1}{2}\omega_3^2 - \sigma_3 \int \omega_1\omega_2 d\tau \tag{46}$$

is an integral,

$$\omega_3 = \sqrt{2C + A\sigma_3\cos^2(\omega_0\tau) + B\sigma_3\sin(\omega_0\tau)\cos(\omega_0\tau)}, \tag{47}$$

where

$$A = \frac{-2\sigma_2^2\cos^2(\omega_0\tau_0)\omega_{10}^2 - 4\cos(\omega_0\tau_0)\omega_{10}\omega_{20}\sin(\omega_0\tau_0)\omega_0\sigma_2}{\omega_0^2\sigma_2} + \frac{-\omega_0^2\omega_{20}^2 + 2\omega_0^2\omega_{20}^2\cos^2(\omega_0\tau_0) + \sigma_2^2\omega_{10}^2}{\omega_0^2\sigma_2}, \tag{48}$$

$$B = \frac{-2\sigma_2^2\cos(\omega_0\tau_0)\omega_{10}^2\sin(\omega_0\tau_0) + 4\cos(\omega_0\tau_0)^2\omega_{10}\omega_{20}\omega_0\sigma_2}{\omega_0^2\sigma_2} + \frac{-2\omega_{20}\omega_0\sigma_2\omega_{10} + 2\omega_0^2\omega_{20}^2\sin(\omega_0\tau_0)\cos(\omega_0\tau_0)}{\omega_0^2\sigma_2}. \tag{49}$$

From (42), the new independent variable is

$$\tau = \int \omega_3 dt. \tag{50}$$

To understand its physical meaning, we recall that to rotate from inertial to body coordinates using the $3 \to 1 \to 3$, $\phi \to \theta \to \psi$ sequence entails the well-known expressions for the components of the vector of the body angular velocity $\boldsymbol{\omega}$ in terms of the Euler angles rates $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$:

$$\omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \tag{51a}$$

$$\omega_2 = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi, \tag{51b}$$

$$\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta. \tag{51c}$$

Thus,

$$\tau = \int \omega_3 = \psi + \int \dot{\phi}\cos\theta dt, \tag{52}$$

so the ω_i-τ dynamics may be viewed as a solution for the *phase space* of the Eulerian system.

The rescription in the Eulerian case possesses an interesting symmetry. In the above discussion, we detected the dynamic rescriptor ω_3 for the pair $(\dot{\omega}_1, \dot{\omega}_2)$, but there are two other possible rescriptions: ω_1 for $(\dot{\omega}_2, \dot{\omega}_3)$ and ω_2 for $(\dot{\omega}_1, \dot{\omega}_3)$.

Finally, we note that the rescriptive gauge transformation *linearized* the Euler-Poinsot equations; observe that (43a) and (43b) are linear, and (43c) is a simple linear quadrature thereof in the variable $z = \omega_3^2$. We shall further dwell upon this finding in Sec. III E.

C. Other common systems exhibiting rescriptive gauge symmetry

The gauged pendulum is a particular case of systems exhibiting rescriptive gauge symmetry. However, there are systems that exhibit rescriptive gauge symmetry, which cannot be rendered periodic after a rescriptive gauge transformation. Generally speaking, such systems cannot be conveniently described using the Hamiltonian formalism, although they do possess integrals. Consider, for illustration, the dynamical equations of two chemical reactants, A and B, whose concentrations evolve according to the bilinear rate law:[14]

$$\frac{d[A]}{dt} = k_1[A][B], \tag{53a}$$

$$\frac{d[B]}{dt} = k_2[A][B]. \tag{53b}$$

These can be de-rescribed using, e.g., $d\tau = [A]dt$, yielding the linear equations

$$\frac{d[A]}{d\tau} = k_1[B], \tag{54a}$$

$$\frac{d[B]}{d\tau} = k_2[B], \tag{54b}$$

so that

$$[B(\tau)] = [B(\tau_0)]e^{k_2\tau}, \quad [A(\tau)] = [B(\tau_0)]\frac{k_1}{k_2}(e^{k_2\tau} - 1) + [A(\tau_0)]. \tag{55}$$

An integral for system (54) is $\mathcal{C} = [A] - (k_1/k_2)[B]$, albeit this is not the Hamiltonian. Consequently, an additional class of systems exhibiting rescriptive gauge symmetry may be written as

$$\dot{q}_i = q_i u_i(\mathbf{p}, \mathbf{q}), \tag{56a}$$

$$\dot{p}_i = q_i u_i(\mathbf{p}, \mathbf{q}), \tag{56b}$$

$$\dot{u}_i = h_i(\mathbf{p}, \mathbf{q}, u_i, \mathbf{y}), \tag{56c}$$

$$\dot{\mathbf{y}} = \mathbf{h}_1(\mathbf{u}, \mathbf{y}). \tag{56d}$$

D. The one-parameter Lie symmetry group

Thus far, we have not explicitly spelled out a relationship between the rescriptive gauge transformation and Lie point-symmetry transformations. This is the purpose of the following discussion.

To keep things simple, assume a 1-DOF gauged pendulum model with a static rescriptor, $u(p,q)$:

$$\dot{q} = pu(p,q), \tag{57a}$$

$$\dot{p} = -qu(p,q). \tag{57b}$$

This set of equations can be analyzed by means of one-parameter groups based upon infinitesimal transformations. We demand the equation to be invariant under infinitesimal changes of the independent variable t, but *without* simultaneous infinitesimal changes of the dependent variables. This leads to the Lie point-symmetry transformation

$$p \to p, \tag{58a}$$

$$q \to q, \tag{58b}$$

$$t \to \tau = t + \epsilon\zeta(p,q). \tag{58c}$$

We now apply (58) on (57) by following these stages. First, we write

$$\frac{dq}{d\tau} = \frac{dq}{dt + \epsilon\left(\frac{\partial\zeta}{\partial q}dq + \frac{\partial\zeta}{\partial p}dp\right)} + O(\epsilon^2) \tag{59}$$

and

$$\frac{dp}{d\tau} = \frac{dp}{dt + \epsilon\left(\frac{\partial\zeta}{\partial q}dq + \frac{\partial\zeta}{\partial p}dp\right)} + O(\epsilon^2). \tag{60}$$

Expanding (59) and (60) into a Taylor series with ϵ as a first-order small parameter, we get

$$\frac{dq}{d\tau} = \frac{dq}{dt} - \epsilon\frac{dq}{dt}\left(\frac{\partial\zeta}{\partial q}\frac{dq}{dt} + \frac{\partial\zeta}{\partial p}\frac{dp}{dt}\right) + O(\epsilon^2), \tag{61}$$

$$\frac{dp}{d\tau} = \frac{dp}{dt} - \epsilon\frac{dp}{dt}\left(\frac{\partial\zeta}{\partial q}\frac{dq}{dt} + \frac{\partial\zeta}{\partial p}\frac{dp}{dt}\right) + O(\epsilon^2). \tag{62}$$

These yield a partial differential equation (PDE) for $\zeta(p,q)$,

$$u^2(p,q)\epsilon\left(p\frac{\partial\zeta(p,q)}{\partial q} - q\frac{\partial\zeta(p,q)}{\partial p}\right) - u(p,q) + 1 = 0; \tag{63}$$

the solution thereof is

$$\zeta(p,q) = -\int^{p} \frac{u(\eta,\sqrt{c-\eta^2}) - 1}{\epsilon\sqrt{c-\eta^2}u^2(\eta,\sqrt{c-\eta^2})}d\eta + c_0 c, \tag{64}$$

where $c=p^2+q^2$ and c_0 is an integration constant. To relate (59) and (60) to the generators of the infinitesimal transformation, we write

$$\tau = t + \epsilon\zeta(p,q) + \cdots = t + \epsilon \mathrm{X}t + \cdots, \tag{65}$$

where the operator X is given by

$$\mathrm{X} = \zeta(p,q)\frac{\partial}{\partial t}. \tag{66}$$

In addition, due to the fact that (57) is autonomous, it will also exhibit Lie point-symmetry with generator

$$\mathrm{X}_1 = \frac{\partial}{\partial t}. \tag{67}$$

Symmetries (66) and (67) form an Abelian Lie algebra $\mathfrak{X}$ with the Lie bracket $[\mathrm{X},\mathrm{X}_1]=0$.

In essence, this symmetry implies that the *direction field* is

$$\frac{dp}{dq} = -\frac{q}{p}, \tag{68}$$

and is therefore homogeneous, that is, invariant under all dilations $(p,q)\mapsto(e^{\lambda}p,e^{\lambda}q)$, $\lambda\in\mathbb{R}$, which holds true for any dynamic or static rescriptive gauge $u(p,q)$. The connection to the rescriptive gauge symmetry can now be easily obtained via Arnold's theorem,[15] stating that if a one-parameter group of symmetries of a direction field is known, the equation $dp/dq=f(p,q)$ can be integrated explicitly. This is obvious for the direction field (68) of the gauged pendulum.

E. Reduction using rescriptive gauge symmetry

It is a well-known fact in dynamical system theory that under certain conditions, systems that exhibit symmetry are also reducible.[16] We shall discuss reduction in the context of rescriptive gauge theory by following a few fundamental steps; ultimately, we will show that rescriptive gauge symmetry allows to reduce classes of nonlinear systems into linear ODEs, solved by simple quadratures.

We begin our quest for the manifestation of reduction in the realm of rescriptive gauges by asking how a rescriptor for a given ODE can be found. We shall then show that the answer to this question is related to a more profound problem: that of *exact linearization* of ODEs, or, as we shall call it for clarity, *global linearization*. We shall dwell upon the latter issue shortly, and will first address the more basic query.

Finding a rescriptive gauge transformation for a given ODE is important, since it may allow quadrature in the modified time scale by reduction into linear forms. Consider, for illustration, the 1-DOF gauged pendulum model

$$\dot{q} = pu(p,q), \tag{69a}$$

$$\dot{p} = -qu(p,q), \tag{69b}$$

which is readily transformed into the ODE

$$\ddot{q} - \frac{\partial u(p,q)}{\partial q}p\dot{q} + u(p,q)q\left[\frac{\partial u(p,q)}{\partial p} + u(p,q)\right]. \tag{70}$$

Thus, any ODE that is written in the form (70) can be transformed into the de-rescribed gauged pendulum $q''+q=0$ using the rescriptive gauge transformation $d\tau=udt$. However, usually the rescriptor, u, cannot be easily found. Consider, for instance, the nonlinear ODE

$$\ddot{q} - \dot{q}^2\cot q + q\sin^2 q = 0, \tag{71}$$

for which the rescriptive gauge transformation

$$d\tau = \sin q\, dt, \tag{72}$$

reveals that (71) is no more than a harmonic oscillator in disguise; viz., $q''+q=0$. However, one cannot determine that $u=\sin q$ by observation. This calls for a more rigorous methodology for finding the rescriptor.

To that end, consider a second-order ODE of the form

$$\ddot{q} + f(q)\dot{q}^2 + b_1 u(q)\dot{q} + \psi(q) = 0. \tag{73}$$

When can this ODE be transformed into the linear form

$$q'' + b_1 q' + b_0 q + c = 0 \tag{74}$$

by a rescriptive gauge transformation

$$d\tau = u(q)dt \tag{75}$$

only? The answer lies in the theory of *exact linearization*,[17] which seeks a transformation rendering a nonlinear ODE amenable for quadrature. We shall prefer the term *global linearization*, emphasizing that this method is conceptually different from the common point linearization. We shall ultimately use global linearization theory to help us track down the rescriptor of a given ODE.

The theory of global linearization suggests that ODEs of the form (73) can be globally linearized by a transformation of the form

$$z = \beta \int u \exp\left(\int f dq \right) dq, \quad d\tau = u(q)dt, \tag{76}$$

where β=const., if and only if (74) can be written in the form

$$\ddot{q} + f(q)\dot{q}^2 + b_1 u\dot{q} + u \exp\left(-\int f(q)dq \right) \times \left[b_0 \int u \exp\left(\int f(q)dq \right) dq + \frac{c}{\beta} \right] = 0. \tag{77}$$

This fundamental result can be adapted to the case in question. In particular, since we are probing the case of rescriptive gauge transformations, we must require that $z=q$, or, in other words, that

$$\beta = 1, \quad u = u(q), \quad f = -\frac{1}{u}\frac{du}{dq}. \tag{78}$$

In our discussion, we allowed u to be a function of both q and p, while Eqs. (76) permit a u that is a function of q only. Thus, we must take $u=u(q)$, as written in (78). Relations (78) modify (77) into

$$\ddot{q} - \frac{1}{u}\frac{du}{dq}\dot{q}^2 + b_1 u\dot{q} + u \exp\left(-\int f(q)dq \right)\left[b_0 q + \frac{c}{\beta} \right] \tag{79}$$

$$= \ddot{q} - \frac{1}{u}\frac{du}{dq}\dot{q}^2 + b_1 u\dot{q} + u \exp\left(-\int \frac{1}{u}\frac{du}{dq} dq \right)\left[b_0 q + \frac{c}{\beta} \right] \tag{80}$$

$$= \ddot{q} - \frac{1}{u}\frac{du}{dq}\dot{q}^2 + b_1 u\dot{q} + u^2 b_0 q + \frac{c}{\beta} = 0. \tag{81}$$

Thus, we have proven that a second-order ODE can be transformed into a linear ODE using a rescriptive gauge transformation (assuming that the rescriptor is a function of the coordinate only) *if and only if* this ODE can be written as

$$\ddot{q} - \frac{1}{u(q)}\frac{du(q)}{dq}\dot{q}^2 + b_1 u(q)\dot{q} + u(q)^2 b_0 q + \frac{c}{\beta} = 0. \tag{82}$$

Equation (82) immediately yields the rescriptor: It is the square root of the coefficient of the coordinate q divided by $\sqrt{b_0}$.

Returning to example (71), we see that it succumbs to the general form (82) by substituting

$$b_1 = 0, \quad b_0 = 1, \quad c = 0, \tag{83}$$

which yields

$$\ddot{q} - \frac{1}{u}\frac{du}{dq}\dot{q}^2 + u^2 q = 0, \tag{84}$$

and it is immediately apparent that the rescriptor is $u=\sin q$, which agrees with (72). As a simple verification, we also note that

$$\frac{1}{u}\frac{du}{dq} = \cot q. \tag{85}$$

Similarly, the chemical rate equations (53), written as the single ODE

$$\frac{d^2[A]}{dt^2} - \frac{1}{[A]}\left(\frac{d[A]}{dt} \right)^2 - k_2[A]\frac{d[A]}{dt} = 0, \tag{86}$$

may be linearized using the transformation $d\tau=[A]dt$, as was done in Sec. III C.

The above process can be repeated for higher-order ODEs as well. The bottom line is that the theory of global linearization is a convenient method for finding a rescriptor of a given ODE, or, in other words, to reduce it into a linear ODE using a rescriptive gauge transformation.

To conclude this section, we shall show that there are well-known ODEs that can be transformed into the reducible form (82) using an additional auxiliary variable transformation. This observation is inspired by Sec. III B, where we have shown that the Euler-Poinsot equations are transformed into a linear from in the independent variable τ using the rescriptive gauge transformation $d\tau=\omega_3 dt$ and the auxiliary transformation $z=\omega_3^2$. For example, consider the ODE:

$$\ddot{q} + q\dot{q} + kq^3 = 0, \quad k = \text{const.} \tag{87}$$

This ODE arises in a few practical problems.[18] To render it globally linearizable using a rescriptive gauge transformation, perform the auxiliary variable transformation $z=q^2$, so the modified system reads

$$\ddot{z} - \frac{1}{2z}\dot{z}^2 + \sqrt{z}\dot{z} + kz^2 = 0. \tag{88}$$

In this form, (88) adheres to ansatz (82), with the rescriptor $u=\sqrt{z}=q$ and $k=b_0$, $b_1=1$, $c=0$. The rescriptive gauge transformation $d\tau=\sqrt{z}dt$ transforms (88) into

$$z'' + z' + 2kz = 0. \tag{89}$$

F. Illustrative examples

We shall now illustrate the rescriptive gauge transformation formalism and the resulting gauged pendulum concept using a few numerical examples.

Example 1: A damped pendulum is a gauged pendulum

Consider the model[19]

$$\dot{q} = up, \tag{90a}$$

$$\dot{p} = -uq, \tag{90b}$$

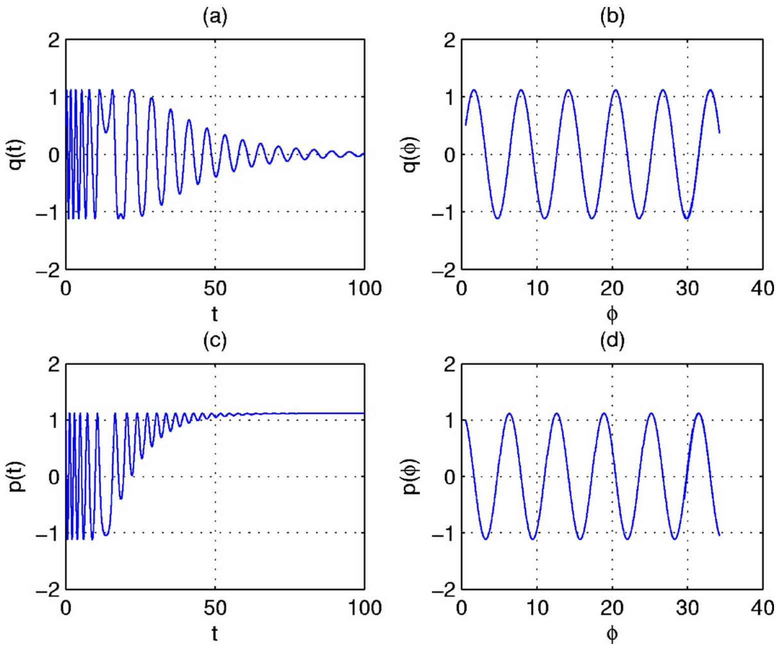

FIG. 1. An exponential decay of a damped nonlinear pendulum can be transformed into harmonic oscillations by a rescriptive gauge transformation.

$$\dot{u} = -\omega_0^2 q - au. \tag{90c}$$

By carrying out the transformation $p=\cos\phi$, $q=\sin\phi$, these equations are immediately recognized as a state-space model for a damped nonlinear pendulum,

$$\ddot{\phi} + \omega_0^2 \sin\phi + a\dot{\phi} = 0. \tag{91}$$

System (90) complies with the gauged pendulum formalism (28); it can be therefore viewed as a *rescribed harmonic oscillator*, revealed by the rescriptive gauge transformation $d\phi = u dt$, so that $\tau = \phi$:

$$q' = p, \quad p' = -q. \tag{92}$$

Obviously, the rescriptor, or gauge velocity, is simply the angular velocity; i.e., $u=\dot{\phi}$. The scalar differential equation for this dynamic rescriptive gauge function [Eq. (90c)] assumes the nonautonomous form

$$u' = -\omega_0^2 \sin(\phi)/u - a. \tag{93}$$

For $a=0$, the rescriptive gauge function does not explicitly depend upon the rescriptor itself, and (93) is easily solved by quadrature:

$$u(\phi) = \sqrt{2\omega_0^2(\cos\phi - \cos\phi_0) + u^2(\phi_0)}. \tag{94}$$

It is interesting to note that under the rescriptive gauge symmetry, the harmonic oscillator and the damped nonlinear pendulum are represented by the *same* mathematical formalism—although for different independent variables—whereas the time flow of these models is completely different. The harmonic oscillator, which is a conservative system, does not have an attractor, since the motion is periodic. The damped pendulum, on the other hand, is a dissipative dynamical systems, in which volumes shrink exponentially, so its attractor has zero volume in phase space. This alleged paradox stems from the fact that the dissipative time flow of the damped pendulum becomes periodic under a change of the independent variable. Thus, an observer measuring the "time" ϕ is bound to observe periodic behavior, while an observer measuring the "true" time t will observe exponential decay.

These observations are demonstrated and validated by means of a numerical integration, comparing the flows of (90) and (92). Figure 1 compares between $q(t)$ [Fig. 1(a)] and $q(\phi)$ [Fig. 1(b)], and between $p(t)$ [Fig. 1(c)] and $p(\phi)$ [Fig. 1(d)], for $a=0.1$, $q_0=0.5$, $p_0=1$, $u_0=5$, $\phi_0=\sin^{-1}q_0=0.5236$.

Example 2: A glimpse of order in the realm of chaos

Consider the dynamical system

$$\dot{q} = yp, \tag{95a}$$

$$\dot{p} = -yq, \tag{95b}$$

$$\dot{x} = \sigma(y-x), \tag{95c}$$

$$\dot{y} = (r-z)x - y, \tag{95d}$$

$$\dot{z} = xy - bz, \tag{95e}$$

where σ, r, and b are constants. Equations (95c)–(95e) are recognized as the Lorenz system, and the entire system (95) complies with the gauged pendulum formalism of Eqs. (25). It shall be thus referred to as the *Lorenz-fed gauged pendulum*.

For certain parameter values and initial conditions, the Lorenz system is known to exhibit chaos. For instance, choosing the parameter values $\sigma=10$, $r=28$, $b=8/3$, the initial conditions $x(0)=10$, $y(0)=10$, $z(0)=10$, and simulating for $t_f=50$ time units, yields the trajectory depicted by Fig. 2.

Let us now examine the time history of p and q, shown in Fig. 3, and ask: Do q and p exhibit chaotic behavior? To answer this seemingly trivial question (without using a comprehensive mapping of the phase space using Poincaré sections), we shall resort to the common "engineering" interpre-

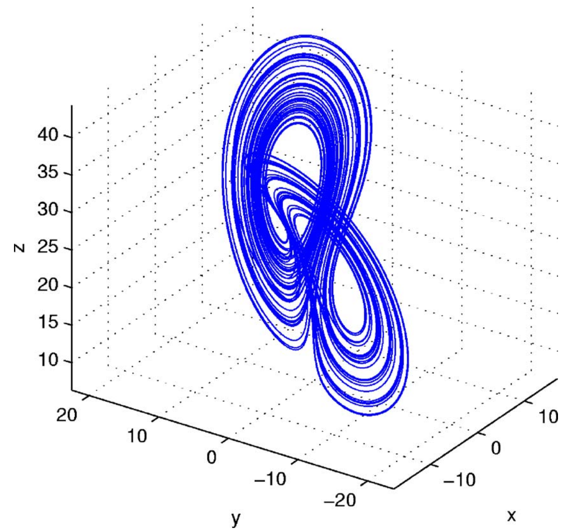

FIG. 2. The Lorenz strange attractor feeding the gauged pendulum.

tation of chaos, although more mathematically rigorous definitions, related to the destruction of KAM tori[20] or the Kolmogorov-Sinai entropy,[21] do exist. As Strogatz says in Ref. 22, "no definition of the term chaos is universally accepted yet, but almost everyone would agree on the three ingredients used in the following working definition." These three ingredients are:

(1) Aperiodicity: Chaos is aperiodic long-term behavior in a deterministic system. Aperiodic long-term behavior means that there are trajectories that do not settle down to fixed points, periodic orbits, or quasiperiodic orbits as $t \to \infty$. (For the purposes of this definition, a trajectory that approaches a limit of ∞ as $t \to \infty$ should be considered to have a fixed point at ∞.)

(2) Sensitive dependence on initial conditions: Nearby trajectories separate exponentially fast, i.e., the system has a positive Lyapunov characteristic exponent (LCE).

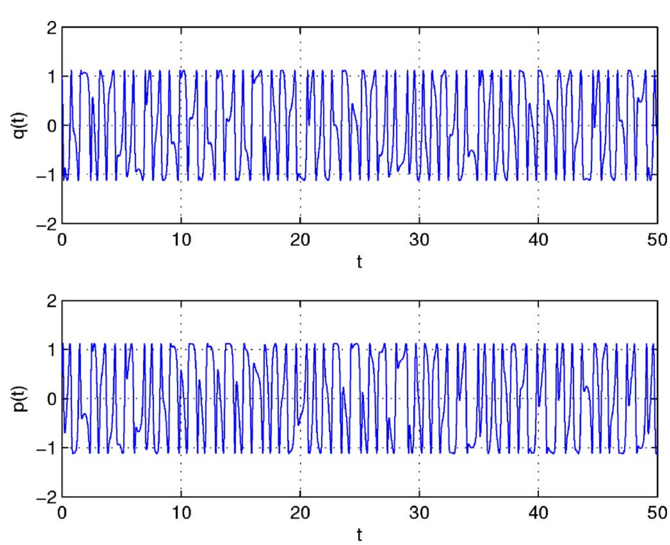

FIG. 3. A seemingly irregular behavior of a gauged pendulum fed by a chaotic process.

(3) Strogatz notes that he favors additional constraints on the aperiodic long-term behavior, but leaves open what form they may take. He suggests two alternatives to fulfill this:

(a) Requiring that there exists an open set of initial conditions having aperiodic trajectories, or

(b) If one picks a random initial condition $x(t_0)=x_0$ then there must be a nonzero chance of the associated trajectory $x(t,x_0)$ being aperiodic.

Returning to Fig. 3, we see that items 1, 3(a), and 3(b) in Strogatz's list are satisfied: p and q exhibit aperiodic behavior, the open set of initial conditions guaranteeing aperiodic trajectories for $\sigma=10$, $r=28$, $b=8/3$ are $x_0, y_0, z_0, p_0, q_0 \in \mathbb{R}\backslash\{0\}$, and hence for randomly selected initial conditions, p and q will be aperiodic. The only remaining test is to calculate the LCEs, denoted by λ_i, $i=1,\ldots,n$. However, as shall be illustrated shortly, calculation of the LCEs may be problematic for system (95).

First, we should note that some authors endorse the calculation of the *maximal* Lyapunov exponent in order to establish the presence of chaos. For example, Ref. 23 states that "it is well known that the ordered or the chaotic property of an orbit is characterized by the largest Lyapunov characteristic exponent." This approach, however, is misleading for system (95). To illustrate this fact, we have calculated the maximal Lyapunov exponent for (95) using the standard method developed by Refs. 24 and 25. The result is depicted by Fig. 4 for an integration period of 12 000 time units. It is seen that the maximal LCE satisfies $\max_i \lambda_i \approx 0.9$, which is the well-known maximal LCE of the Lorenz system. Hence, according to the rationale of Ref. 23, system (95) is chaotic—or is it?

For a more rigorous analysis, the entire spectrum of LCEs should be examined. Since the LCE spectrum of the Lorenz system is well known {the phase-space contraction satisfies the relation $\Sigma_i \lambda_i = \nabla \cdot [\dot{x}, \dot{y}, \dot{z}] = -(\sigma+b+1) = -13.667$}, let us concentrate on the additional LCEs contributed by p and q. A magnified view of these LCEs is shown in Fig. 5. One of these LCEs is smaller than zero, while the other one assumes the value of 4×10^{-5}, which allegedly indicates that the additional states are also chaotic.

However, this is a mere illusion resulting from the fact that the calculation process of the LCEs is affected by the truncation and round-off errors of the numerical integration routine used to simultaneously integrate the extended phase space of the original and linearized systems. (This causes the Lyapunov exponents themselves to exhibit a chaotic behavior; most high-order integrators are chaotic maps, as pointed out in Ref. 26. This may be viewed a manifestation of the uncertainty principle.) One may view this phenomenon as *pseudochaos*;[27] the truth regarding "chaos" in system (95) can be plainly revealed by realizing that (95) complies with the gauged-pendulum formalism, and can hence be subjected to a rescriptive gauge transformation of the from $d\tau = y dt$. This transformation will transform (95a) and (95b) into $q' = p, p' = -q$, which is an integrable system and hence *cannot* exhibit chaos. This observation is illustrated in Fig. 6, showing plots of q and p as a function of τ. Thus, in contrast to

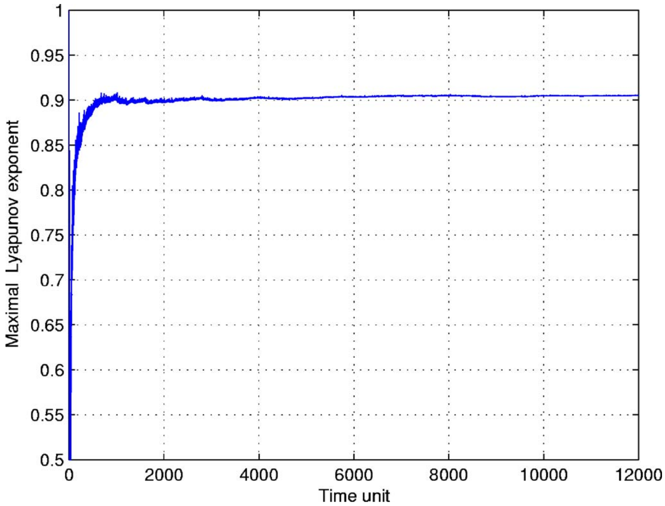

FIG. 4. The maximal Lyapunov characteristic exponent for a Lorenz-fed gauged pendulum system.

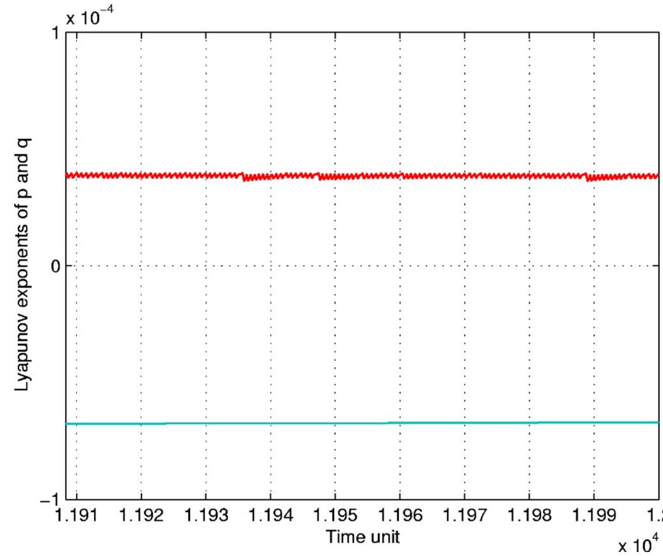

FIG. 5. The Lyapunov exponents contributed by the states p and q.

the prediction of the common engineering interpreting of chaos and the chaos detection tools thereof, the rescriptive gauge transformation shows that the temporal behavior of signals cannot always be used to predict the presence of chaos. This observation calls into being the concept of *partial chaos*,[28] meaning that in a given system, both chaotic and regular signals may coexist, even if the chaotic states overshadow the regular behavior of the other states.

Another important conclusion concerns the system *observables*. Observables, or outputs, is a subset of state variables, i.e., $\mathbf{z}$, $\dim \mathbf{z}=l \leq \dim \mathbf{x}=n$, determined by the *output map* $\mathfrak{O}:\mathbb{R}^n \to \mathbb{R}^l$, such that $\mathbf{z}=\mathfrak{O}(\mathbf{x})$, and an *observation scale* $\mathfrak{T} \in \mathbb{R}$, such that $\mathbf{z}:\mathbb{R} \to \mathbb{R}^l$. If $\mathfrak{T}=t$, the observation process is *temporal* and the observable scale is merely the time. Our simple example shows that temporal observations may be misleading when used to detect chaos, even when using a seemingly rigorous test such as the LCE spectrum. A fictitious observer using $\mathfrak{T}=\tau$ as the scale would have not suspected that the Lorenz-fed gauged pendulum is a chaotic process.

We further conclude that rescriptive gauge transformations may be used to isolate self-similarities of a dynamical systems. In our example, the Lorenz system remains scale invariant; i.e., its Hausdorff dimension does not depend on the scale. However, the Lorenz-fed pendulum is *not* scale invariant, and hence is a regular process in disguise.

Example 3: Stochastic signals, coding, and Kolmogorov complexity

The preceding example illustrated the fact that the gauged pendulum concept may be used to order pseudochaotic behavior. This is, in fact, only an understatement of the potential of rescriptive gauge theory; this theory can be used not only for ordering pseudochaotic signals, but moreover, to transform seemingly stochastic signals into deterministic ones.

Our final example is therefore concerned with illustrating how rescriptive gauge symmetry, and in particular a simple gauged pendulum, may be used to establish some key ideas in modern information and coding theory through the well-known notion of Kolmogorov complexity.

The Kolmogorov complexity (also known as Kolmogorov-Chaitin complexity, stochastic complexity, and algorithmic entropy) of an object is a measure of the computational resources needed to specify the object.[29–31] In other words, the complexity of a string is the length of the string's shortest description in some fixed description language. It can be shown that the Kolmogorov complexity of any string cannot be too much larger than the length of the string itself. Strings whose Kolmogorov complexity is small relative to the string's size are not considered to be complex. The sensitivity of complexity relative to the choice of description "language" is what the current example is about. To that end, consider the gauge pendulum

$$\dot{q}=wp, \quad \dot{p}=-wq, \tag{96}$$

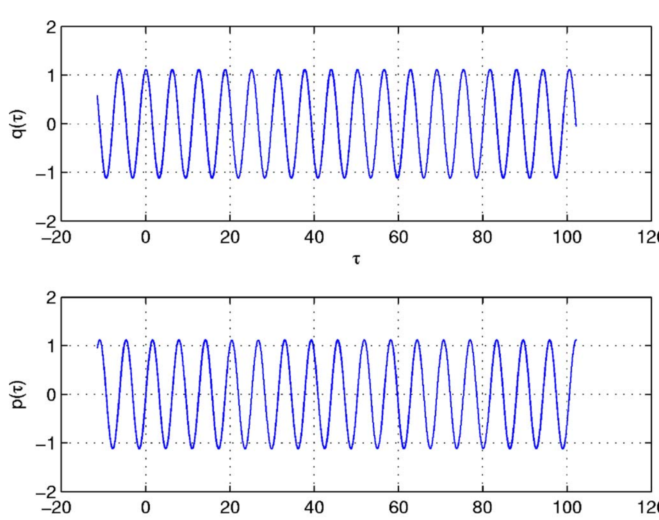

FIG. 6. The seemingly irregular behavior of the Lorenz–fed gauged pendulum, shown in Fig. 3, can be regularized into harmonic oscillations by a rescriptive gauge transformation.

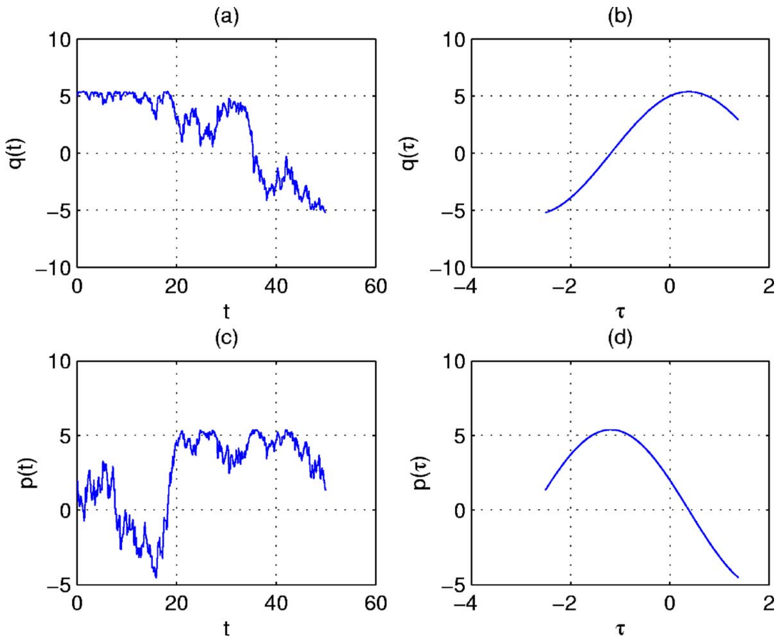

FIG. 7. “Stochastic” signals transformed into harmonic oscillations by a rescriptive gauge transformation.

where here the rescriptor w is a *band-limited white noise*; that is, a white noise going through a zero-order hold with some sampling frequency T_w and power spectral density W. Model (96) can be de-rescribed by $d\tau = wdt$.

Let us compare the representation of the “strings” q and p using the “languages” t, time, and τ, a random walk obtained by integrating w (i.e., a stochastic signal in its own right). This comparison is depicted in Fig. 7 for $T_w = 0.1$ time units and $W = 0.1$. Figure 7(a) shows the signal $q(t)$, which should be compared to the signal $q(\tau)$, shown in Fig. 7(b). Similarly, compare $p(t)$ [Fig. 7(c)] to $p(\tau)$ [Fig. 7(d)].

Although $q(t)$ and $p(t)$ seem stochastic and therefore Kolmogorov-complex in the “language” t, their alleged complexity vanishes when the “language” τ is used, and the stormy stochasticity vanishes into harmonic oscillations, implying much reduced Kolmogorov complexity. This phenomenon has practical value in terms of coding theory: Signals may be coded using the “code” t and decoded using the “key” τ.

IV. SUMMARY AND CONCLUSIONS

This paper described how gauge theory can be adapted for finite-dimensional dynamical systems. We have defined gauge symmetry in the following context: Rescriptive gauge symmetry results from an action of a one-parameter Lie group, yielding an Abelian Lie algebra. A rescriptive gauge symmetry transformation is then an infinitesimal change of the independent variable, which renders the system integrable via reduction.

The gauge conversation leads to a few practical conclusions. We first note that gauge symmetry is ubiquitous in a myriad of scientific fields. Gauge theory for finite-dimensional system may be thus viewed as a generalization of dynamical systems theory into the realm of group theory, unifying various physical phenomenon into simple generating models.

Furthermore, the gauge-theoretic tools may be used to improve our understanding of chaos, randomness and their inter-relations. We discussed a few simple examples showing how a change of scale can lead to pattern evocation in seemingly chaotic and/or stochastic systems.

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